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A TEXT BOOK OF ELEMENTARY MECHANICS,

FOR THE USE OF

COLLEGES AND SCHOOLS.

BY

EDWARD S. DANA,

("*Professor of Physics in Yale College.*")

TWELFTH EDITION.

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PREFACE.

The writer has been induced to prepare this Text Book because of his inability to find, among the many excellent works on Mechanics, one that was thoroughly adapted to his special wants, and because it has seemed to him probable that similar needs must have been felt by other instructors.

The chief aim has been to present the fundamental principles of the subject in logical order, and in as clear, simple, and concise a form as possible, yet without any sacrifice of strict accuracy. For the sake of making the portions of the subject, which necessarily involve some difficulty, more intelligible to beginners, and also to increase the interest of the general principles demonstrated by showing something of their practical bearings, simple illustrations have been introduced rather more fully than usual; these are sometimes given in a few words, sometimes in more extended form. This has led to a slight expansion of the book in size, but does not proportionately increase the time required to master it. The general scheme is not more extended than the subject demands, and, if time is limited, the instructor can readily select those articles whose omission will not interfere with the completeness of the study. [Some

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ELEMENTARY MECHANICS.

INTRODUCTION.

1. Matter. Matter is the substance of which bodies re composed; it is that which may be apprehended by he senses, and which may be acted upon by force.

2. Body-Particle. A *body* is any portion of matter which is bounded in every direction. A *material particle* is a body of dimensions so small that it is unnecessary o consider the differences in position or motion of its lifferent parts.

In many cases the differences in the relations of the parts of an extended body are, in like manner, left out of account, it being considered as a single unit, and then the body is treated as a particle.

3. Molecule. The smallest portion into which a given kind of matter can be conceived to be divided, without a loss of its properties, is called the *molecule.* The molecule is an ideal unit, the existence of which is believed to be proved by experiment, although it cannot be by direct observation. The smallest portion of matter, obtained by any method of mechanical subdivision, would consist of a large number of molecules.

According to the conclusions of Sir William Thomson, f a drop of water were to be magnified to the size of the earth, the molecules, of which it is made up, would be coarser than fine shot and probably finer than cricket-balls.

4. Physical Science. All changes which involv material body, either as a whole or with respect to relations of its molecules, are considered under the h of Natural Philosophy, or *Physical Science*. Thus, fall of a body to the earth; the flight of a rifle-ball; ring of a bell; the melting of iron, and its contract or expansion on change of temperature, its magnet tion—these and all other analogous phenomena are cluded under Physical Science.

5. Atom. Every molecule is supposed to be made of one or more indivisible units called *atoms* (ἄ p and τέμνω, to divide). Thus, the smallest conceiv particle, or molecule, of salt, possessing all the pro ties of the mass, is believed to consist of two dissim atoms, one of the metal sodium, the other of the chlorine.

6. Chemistry. All phenomena which result in a arrangement of the atoms and a consequent chang the molecules of a body—that is, a loss of identity o substance involved—belong to *Chemistry*. For exa the change of ice to water, or of water to steam, invo no change in the molecules but only in their mutual lations and position, hence these phenomena belong Physical Science; but when a rearrangement of the at takes place and the water is thus decomposed into constituent gases, hydrogen and oxygen, this last *chemical* change.

The molecule is the physical unit; the atom is chemical unit.

7. States of Matter. Matter may exist in three di ent states: the *solid*, *liquid*, and *gaseous* states.

The SOLID is characterized by a greater or less d

rigidity. The molecules are bound together by the ›lecular force of attraction, called cohesion, and hence ›olid body tends to retain its own shape.

The LIQUID is characterized by its mobility; the mole-les are free to move about each other, and the liquid kes the shape of any containing vessel.

The GAS is characterized by its tendency to indefinite pansion. The molecules are believed to be in rapid ›tion and constantly coming into collision and then pelling one another, so that a gas tends to occupy a eater volume, and hence exerts pressure on the sides any vessel in which it is confined.

The term *fluid* is sometimes employed to include both ›uids and gases.

Many substances may under varying conditions exist the three different states: this is illustrated by the se of water, which is a solid—ice—below the freezing ›int, a liquid at ordinary temperatures, and a gas—steam at high temperatures.

8. **Properties of Matter.** All forms of matter possess e essential properties of *extension, impenetrability,* and *ertia.*

(1) EXTENSION: Every body occupies a definite por-›n of space; that is, it has length, breadth, and thick-ss.

(2) IMPENETRABILITY: Two forms of matter cannot upy the same space at the same time.

(3) INERTIA: Matter has no power to change its state of motion or rest, hence it offers an apparent stance to a force tending to change its state. This is her explained in a subsequent article (67).

Other properties of matter are—

›ROSITY: The molecules, of which a given body is

supposed to be made up, are believed to be separated from one another by a greater or less space. In addition to these true or physical pores, most bodies exhibit also visible open spaces, or sensible pores, as those of a sponge.

COMPRESSIBILITY : A body may be made by pressure to occupy a smaller space ; this is a direct consequence of its porosity.

DIVISIBILITY : A given kind of matter admits of being divided into a very great number of parts.

ELASTICITY : A body, whose shape has been altered by a force acting on it, tends to regain its shape when the force ceases to act. Solids vary widely in elasticity : for example, compare lead and steel, or clay and ivory. Liquids and gases are perfectly elastic.

---

10. MECHANICS is that branch of Physical Science which considers the motion and equilibrium of bodies. Corresponding to the three states of matter, the subject of Mechanics is divided into

(1) Mechanics of SOLIDS.

(2) Mechanics of LIQUIDS, including Hydrostatics and Hydrodynamics.

(3) Mechanics of GASES, or Pneumatics.

The first of these three divisions, which forms the subject of this text-book, is further divided into three parts, *Kinematics*, *Dynamics* or *Kinetics*, and *Statics*.

KINEMATICS * includes the discussion of abstract motion; that is, of the motion of bodies without reference to their mass (quantity of matter), or to the force

* From the Greek κίνημα, motion.

forces which cause their motion. To the idea of space, involved in Geometry, it adds that of time.

DYNAMICS,* or KINETICS,† embraces the discussion of the action of a force, or of forces, in producing the motion of bodies of known mass.

STATICS‡ discusses the action of forces upon bodies in so far as they hold the body acted upon at rest; that is, in equilibrium.

* From the Greek δύναμις, power.
† From κινέω, to move.
‡ From στατικός (ἵστημι), causing to stand.

METRIC SYSTEM.

UNITS OF LENGTH.

		ENGLISH UNITS.	
Kilometer........	1000 meters.	3280.9 feet.	.62137 mile.
Meter............	———	39.37 inches.	3.281 feet.
Decimeter........	.1 meter.	3.937 "	.3281 "
Centimeter.......	.01 "	0.3937 "	.0328 "
Millimeter (*mm*)..	.001 "	0.0394 "	.00328 "

UNITS OF VOLUME.

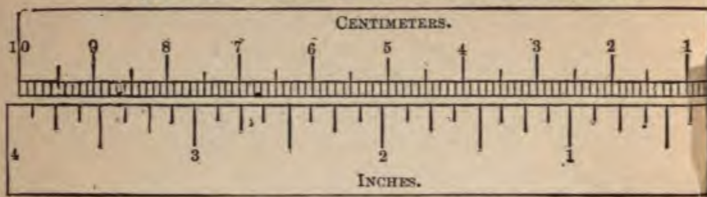
		DRY MEASURE.	LIQUID MEASURE.
Kiloliter (or Stere)	1 cub. meter,	1.308 cub. yds.	264.17 galls.
Liter......... ...	1 cub. decimeter,	0.908 quarts,	1.0567 quarts.
Milliliter..........	1 cub. centimeter,	0.061 cub. in.	0.27 fl. dr'hm.

UNITS OF WEIGHT.

	VOLUME OF WATER GIVING THE WEIGHT.	AVOIRDUPOIS MEASURE. (1 lb. = 7000 grains.)	
Kilogram....	cub. dec'm'r or liter.	15432.3 grains.	2.2046 lbs.
Gram........	cub. centimeter.	15.432 "	0.0022 "
Milligram....	cub. millimeter.	0.0154 "	0.0000022 "

METRIC SYSTEM.

$\frac{1}{10}$ Meter = 1 Decimeter = 10 Centimeters = 100 Millimeters.



4 inches, each divided into eighths.

The *Meter* is the length at 0° C (temperature of melting ice) of a certain platinum bar kept at Paris. It was intended to be (and is very nearly) equal to one forty-millionth part of the earth's circumference about the poles. It is the only arbitrary unit of the *Metric System*, since all the other units of weight, etc., are directly derived from it.

UNITS OF LENGTH. The meter is divided into 10 *decimeters*, into 100 *centimeters*, into 1000 *millimeters*. Also, 10 meters=1 *dekameter*, 100 meters=1 *hectometer*, 1000 meters=1 *kilometer*. (The same prefixes are used in a case of the other units, with a similar signification.)

The approximate value of some of these units are as follows:—The meter is a little longer than the English yard; it is very nearly equal to 3 feet $3\frac{3}{8}$ inches. The millimeter is a little less than .04 of an inch, or 1 inch is a little more than 25 millimeters. (See figure above.) The kilometer is about $\frac{5}{8}$ of a mile.

UNITS OF SURFACE. The squares of the units of length are taken as the units of surface. The principal units are the *centare*, or square meter; the *are* (=100 square meters); and the *hectare* (=10,000 square meters); the hectare is equal to 2.47 acres.

UNITS OF VOLUME. The cubes of the units of length are taken as the units of volume or capacity. The principal units are the cubic meter or *stere*, equal to 1.3 cubic yards; the cubic decimeter or *liter*, which is a little larger than a wine quart; and the cubic centimeter.

UNITS OF WEIGHT. The weights of the units of volume of water (at 4°C = 39°.2 F when it has its greatest density) are taken as the units of weight. The principal units of weight are the *kilogram* (or kilo), which is the weight of a liter or cubic decimeter of water at 4° C; it is equal to 2.2 pounds; and the *gram*, which is the weight of the cubic centimeter of water at 4° C; it is equal to about 15 grains.

The equivalents of the important metric units are given more exactly on the preceding page.

CHAPTER I.—KINEMATICS.

Motion and Rest—Kinds of Motion.

11. Motion. A body is said to move when, in successive intervals of time, it occupies different positions with reference to some other body considered to be at rest.

The terms *motion* and *rest* are simply relative, for the state of any body in this respect can be judged of only by comparing it with some other body or bodies. For example, the objects on the deck of a steamboat may be at rest with reference to each other and to the boat, while they are in motion as regards the neighboring shore. Again, two trains moving side by side at the same speed may seem to a passenger on either to be at rest, and are actually so as regards each other, while they are in rapid motion as regards the ground over which they are passing.

As the term rest is ordinarily employed in Mechanics, the earth is used as the basis of comparison, and in this sense bodies are said to be at rest which do not move with reference to it, as, for example, the buildings in a city. It is to be remembered, however, that the earth itself and hence all objects upon it are really moving very rapidly through space. In fact we know nothing of *absolute* rest, for all bodies of which we have any knowledge are in motion.

Further than this, there is reason to believe that, independent of the motion of the bodies themselves, the

molecules which make them up have also in all cases a very rapid vibratory motion of their own. Motion is then the actual state of matter so far as we know it, while the rest we observe is only apparent.

12. Kinds of Motion. With respect to its direction, a body may have either motion of translation or of rotation. With respect to its rate, the motion may be uniform or varied.

13. Motion of Translation. If the motion of a body is such that every point in it has the same velocity, and every straight line in it remains parallel to itself, the body is said to have *motion of translation.* This is illustrated by the motion of a sled down a hill, or that of the body of a carriage.

The motion of translation of a particle may be either (1) rectilinear—that is, in a straight line—or (2) curvilinear, in a curved line.

14. Motion of Rotation. A body is said to have *motion of rotation,* or simply to rotate, when it moves about an axis so that the different particles describe concentric circles around it, their velocity increasing with their distance from the axis. This is illustrated by the turning of a wheel on its axle, or the spinning of a top.

15. A body may at the same time have both kinds of motion. For example, the wheel of a carriage rotates about its axle, and also moves forward—that is, has motion of translation—with the rest of the vehicle; if the wheel is blocked, as in descending a steep hill, then it has motion of translation only. Again, the earth has a motion of rotation about its axis and also of translation in its orbit about the sun.

In the statements which follow in regard to the motion

of bodies, motion of translation without rotation is always to be understood unless it is distinctly stated otherwise.

As explained in Art. 2, the term body may be used instead of particle, when the body is considered as a unit, any distinction between the position or motion of the different parts being left out of account; in this sense the term body is employed in the following articles.

16. Path of a Particle or Body. The *path* of a particle, or trajectory as it is sometimes called, is the continuous line, either straight or curved, which it describes as it moves. By the path of a body is ordinarily meant the line described by some definite point in it, usually the centre of gravity, or the geometrical centre.

Uniform Motion.

17. Uniform Motion. The motion of a body is said to be *uniform* if it moves over equal spaces in equal successive intervals of time, however small these be taken. The *velocity*, or rate of motion, is then said to be *constant.*

Constant velocity, or the velocity of a body moving uniformly, is measured by the number of units of linear space passed over in the unit of time.

The UNIT OF SPACE commonly employed is the *foot*, and of time the *second.* The UNIT OF VELOCITY is then a velocity of one foot per second; this is a compound unit sometimes called a foot-second. Thus a constant velocity of 10 would mean that the body passed over 10 feet in each successive second.

Other units of space and time are also not infrequently employed: we speak of the velocity of the earth in its orbit as 19 miles per second; of a train as so many miles an hour, and so on. If the metric system is made use of, the units of distance belonging to it must be taken; that is, the millimeter, meter, kilometer, etc.

18. Constant Angular Velocity. The statement in the preceding article has reference only to linear velocity. When, however, a body rotates on an axis and each particle describes a circle about it, it is often convenient to have an expression for the angular velocity.

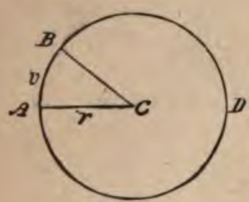


FIG. 1.

The angular velocity of a body, rotating uniformly about an axis, is measured by the angle described in the unit of time by a radius moving in a plane perpendicular to the axis of rotation. This angle, as ACB (Fig. 1), is expressed not in degrees but in circular measure; that is, by the ratio of the arc to the radius $\left(\frac{AB}{AC}\right)$. This angular velocity is usually represented by the letter ω. Hence

$$\omega = \frac{AB}{AC}.$$

If $AC = r$, and $AB = v =$ the linear velocity, then

$$\omega = \frac{v}{r}, \text{ and } \therefore v = \omega r.$$

It is evident that the angular velocity is constant for all parts of a body rotating uniformly, but the linear velocity increases directly with the distance from the axis.

19. Space passed over in Uniform Motion. If a body moves uniformly for a time t, with a velocity v, the space, or distance (s), passed over is equal to *the product of the time and velocity :*

$$s = vt;$$

also,
$$v = \frac{s}{t}, \text{ and } t = \frac{s}{v}.$$

20. Geometrical Representation of Velocity. The velocity of a body may be represented geometrically by a straight line, whose direction is the direction of the motion, and whose length is taken proportional to the velocity. Thus (Fig. 2), if the motion of a particle be

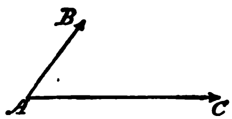


FIG. 2.

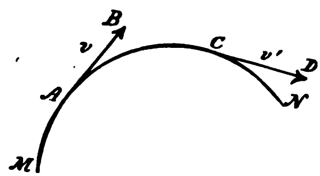


FIG. 3.

in the direction from A toward C with a velocity of 20, and in another independent case from A toward B with a velocity of 10, then these lines, if proportional to 20 and 10 respectively, may be taken as representing these velocities geometrically; here obviously $AC = 2AB$.

If the particle moves in a curved path, as from M toward N (Fig. 3), its velocity at any points, as A and C, will be represented by tangents at these points, AB and CD, whose lengths are proportional to the velocities respectively.

21. Geometrical Representation of the Space passed over in Uniform Motion. The space passed over by a

body moving uniformly for a given time may be rep sented by the area of a rectangle, whose adjacent si are taken proportional respectiv to the time and velocity, each terms of its own unit. Thus, s pose a body to move for t seco with the constant velocity v; AB (Fig. 4) be taken proportio to t, and BC to v. Then, since (19)

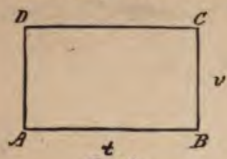


Fig. 4.

$$s = vt, \text{ and } \textit{area of rectangle} = BC.AB,$$

the space is proportional to, or, in other words, is rep sented by, the rectangle.

This principle, which is of interest chiefly from part it takes in a subsequent demonstration (25), me simply that the relation of the area of the rectangle its sides is the same as that of the space in unif motion to the velocity and time.

EXAMPLES.

I. *Uniform Motion of Translation or Rotation.* Articles 17

1. A body travels 30 feet per second: How far will it go day of 24 hours?

2. A velocity of 30 miles per hour corresponds to a rate of many feet per second?

3. A man walks uniformly 4 miles per hour: (*a*) How n feet does he go in a second? (*b*) How many yards in a minut

4. Two bodies start from the same point in opposite direct the one moves at a rate of 11 feet per second, the other at a of 15 miles per hour: (*a*) What will be the distance between t at the end of 8 minutes? (*b*) When will they be 825 feet apa

5. How far will the bodies in the preceding example be a at the end of the same time, if they move in the same directi

6. Two bodies, starting from the same point, move along

at right angles to each other, the first at the rate of $4\frac{1}{8}$ feet per second, the second at a rate of 200 yards per minute: How far will they be apart at the end of an hour?

7. Suppose the earth travels in its orbit 600 million miles in $365\frac{1}{4}$ days: What velocity has it, expressed in miles per second, supposing that the motion is uniform?

---

8. What is the linear velocity of a point on the equator due to the earth's rotation?—take the equatorial radius as 4000 miles.

9. What is the linear velocity of a point on the earth at latitude 60° from the same cause?

10. If the linear velocity of a point at the equator, due to the earth's rotation, is v, show that the velocity at any latitude (l) is equal to $v \cos l$.

11. What is the *angular* velocity of the earth's rotation per second?

12. (*a*) What is the angular velocity of the fly-wheel of an engine, 6 feet in diameter, if it makes 40 revolutions in a minute? (*b*) What is the linear velocity of a point on the circumference?

13. (*a*) What is the angular velocity of a buzz-saw, having a radius of 2 feet, if it makes 100 revolutions per second? (*b*) How far (in miles) will a point on the circumference travel in a working day of 10 hours?

14. The angular velocity of a wheel is $\frac{5}{8}\pi$ per second: What is the linear velocity of points at distances of (*a*) 2 feet, (*b*) 4 feet and (*c*) 10 feet from the centre?

Varied Motion—Acceleration.

22. Varied Motion. The motion of a body is said to be *varied*, and its velocity is called *variable*, if it moves through unequal spaces in equal successive intervals of time. The motion (supposed to be continuous) is said to be *uniformly varied* if the velocity (1) increases or (2) decreases by the same amount in equal successive intervals of time, however small these be taken.

In the first case the motion is *uniformly accelerated*, as the motion of a stone falling toward the earth; in the second case it is uniformly *retarded*, as that of a stone thrown vertically upward.

The velocity of a body, if variable, is measured at any instant by the distance through which the body would pass in the following unit of time, if the motion were to continue uniformly through that time at the same rate.

Thus, we speak of the velocity of a railroad train as being at a certain instant 25 miles per hour, meaning that, if the rate were to be kept up uniformly for the hour following, the train would pass over 25 miles. We know, however, that the supposition will not in fact be realized. Again, the velocity of a falling body, at a certain instant, may be said to be 64 feet per second; and by this is meant that, if it should move uniformly for the next second at the rate it has at the instant under consideration, it would pass over 64 feet. But in fact its velocity is constantly increasing, and it will actually fall through a space greater than 64 feet.

23. Acceleration. If the motion of a body is uniformly accelerated, the equal increment of velocity for each succeeding unit of time—the second—is called the *acceleration;* it is *the rate of change of velocity.*

For example, a body falling freely from rest toward the earth acquires a velocity of about 32 feet per second at the end of 1 second, at the end of 2 seconds its velocity is $(32) + 32$, of 3 seconds it is $(32 + 32) + 32$, and so on. In other words, whatever the previous velocity it may have at any instant, in the second following its velocity is increased by about 32 feet. This increment of velocity of 32 feet-per-second per second (as it should

be expressed in full) is called the acceleration due to gravity, and is denoted by the letter g. In general the acceleration due to the action of any force (as explained in 60) is expressed by the letter f.

It is explained in a following Art. (64, p. 61) that the value of g varies slightly for different points on the earth's surface, being greatest at the poles and decreasing toward the equator, where its value is least. The value for New York is about 32.16 (sometimes called $32\frac{1}{6}$). It also diminishes as the distance from the surface of the earth increases.

It is also explained in article 65, p. 62, that this acceleration due to gravity is the same, at one place, for all bodies, whatever their mass; that is, a bullet and a feather will fall in the same time to the earth from a given point, and acquire the same velocity, if the resistance of the air is eliminated. (Read articles 64, 65, 250.)

If the motion of a body is uniformly retarded, the term acceleration is also employed to indicate the equal loss of velocity for each succeeding second, but it has then a negative sign, as having a direction opposite to that of the initial velocity of the body. This is true of a body thrown up from the earth, or of a body projected on a rough horizontal plane and retarded by friction.

24. Velocity acquired in Uniformly Accelerated Motion. Since the acceleration (f) of a body is the increment of velocity for each successive second, it is clear that, if the body starts from rest, its velocity (v) at the end of t seconds will be equal to ft. That is:

$$v = ft; \text{ for a falling body } v = gt.$$

25. Space passed over in Uniformly Accelerated Motion. *The space passed over by a body, starting from rest and moving with uniformly accelerated motion, is equal*

to one half the product of the acceleration into the square of the time.

$$s = \tfrac{1}{2}ft^2;\ \text{for a falling body}\ s = \tfrac{1}{2}gt^2.$$

Let AB (Fig. 5) be taken proportional to the time (t) and BC, at right angles to it, proportional to the velocity (v) acquired in this time, and connect AC; it will be shown that the space passed over by the body is represented by the area of the triangle ABC.

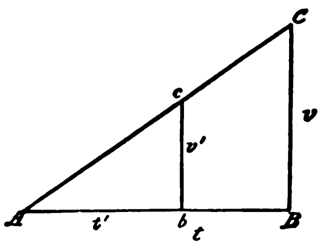


FIG. 5.

First, it is necessary to show that if Ab represents any other time (t') in terms of the same unit, then the corresponding perpendicular bc represents the velocity (v') acquired in this time. For (24)

$$v = ft, \text{ and } v' = ft',$$

$$\therefore f = \frac{v}{t} = \frac{v'}{t'}.$$

Also,
$$\frac{BC}{AB} = \frac{bc}{Ab}, \text{ but } \frac{BC}{AB} = \frac{v}{t},$$

$$\therefore \frac{bc}{Ab} = \frac{v'}{t'}.$$

But, by supposition, Ab represents t', hence bc must

represent v'; that is, the corresponding velocity acquired in this time.

Again, let the time (t) be divided into any number of equal parts represented geometrically by Ab', $b'b''$, etc. (Fig. 6). Erect the perpendiculars $b'c'$, $b''c''$, etc.; by the preceding paragraph, these perpendiculars will represent geometrically the velocities acquired at the end of these times taken from the beginning. Now suppose (1) that the body moves *uniformly* for each of these portions of time with the velocity it has at the beginning of that interval, and (2) with that acquired

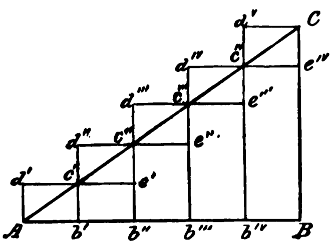


FIG. 6.

at the end. That is, on the first supposition, it moves for the time Ab' with the velocity 0; for the time $b'b''$ with the constant velocity $b'c'$; for the time $b''b'''$ with the velocity $b''c''$, and so on. Then, by Art. 21, the sum of the interior rectangles $0.Ab'$, $b'e'$ $(=b'b''.\ b'c')$, $b''e''$, and so on, will represent the whole space passed over on this first supposition.

On the second supposition the body moves for the time Ab' with the constant velocity $b'c'$; for the time $b'b''$ with the velocity $b''c''$; for the time $b''b'''$ with the velocity $b'''c'''$, etc. Then, in this case, the total

space passed over will be the sum of the exterior rectangles $d'b'$ $(= Ab \times b'c')$, $d''b''$, $d'''b'''$, and so on.

It is obvious that the space represented by the sum of the interior rectangles is less than the true space passed over by the body, and that represented by the sum of the exterior rectangles is greater than the true space; and each differs, by a series of small step-like triangles, from the area of the triangle. Now if the number of intervals into which t is divided be increased indefinitely, and consequently the length of each be indefinitely diminished, and the same construction as that above supposed be carried through, then the sums of the interior and exterior rectangles will approach the area of the triangle as their limit. But the spaces passed over by the body, upon the two suppositions made, also approach the true space (corresponding to a continual and unbroken increase in velocity) as their limit. But when two sets of variable quantities, which are always equal, simultaneously approach their limits, these limits are equal. Therefore

The true space is geometrically represented by the area of the triangle.

$$\therefore\ s = \tfrac{1}{2}BC.AB = \tfrac{1}{2}vt = \tfrac{1}{2}ft^2.$$

26. Average Velocity. From the preceding article $s = \frac{1}{2}vt$, and $\frac{1}{2}v = \frac{s}{t}$. This value of the velocity, obtained by dividing the whole distance by the time, is called the *average velocity*. In the case of uniformly accelerated motion, the average velocity is equal to one half the final velocity acquired; or, in other words, the space passed over is equal to one half the product of this final velocity into the time. This is represented

geometrically by Fig. 7, where, if $BC = 2BE$, it is seen that the areas of the triangle ABC $(= \frac{1}{2}.vt)$ and of the rectangle $ABED$ $(= \frac{1}{2}v.t)$ are equal.

The term average velocity is also employed, in the case of varied motion in general, to denote the result obtained by dividing the whole space by the time. For example, if a train traverses 100 miles in 4 hours, its average velocity is said to be $\frac{100}{4} = 25$ miles per hour,

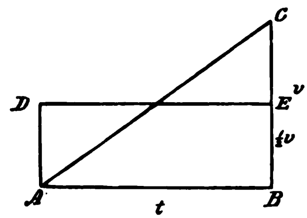


Fig. 7.

although its actual velocity may have varied through very wide limits during the time.

27. Formulas for Accelerated Motion. The results of articles 24 and 25 give

$$v = ft; \qquad \text{for a falling body } v = gt. \qquad (1)$$

$$s = \tfrac{1}{2}ft^2; \qquad \text{" " " } s = \tfrac{1}{2}gt^2. \qquad (2)$$

Therefore, eliminating t from (2),

$$\left.\begin{aligned} s &= \frac{v^2}{2f} \\ \text{or } v^2 &= 2fs \end{aligned}\right\}; \text{ for a falling body } \left.\begin{aligned} s &= \frac{v^2}{2g} \\ v^2 &= 2gs \end{aligned}\right\}. \qquad (3)$$

These three equations give the most important relations for bodies starting from rest and moving with uniformly accelerated motion. From them we see that

(1) The velocity acquired is proportional to the time.

(2) The space is proportional to the square of the time.

(3) The space is proportional to the square of the velocity acquired.

28. From equation (2) of the preceding article it is seen that the space described in the first second from rest is equal to one half the acceleration. Also, the spaces described in 1, 2, 3, 4, etc., seconds are, by the formula:

1 sec.	2.	3.	4.	
$\frac{1}{2}f$,	$\frac{4}{2}f$,	$\frac{9}{2}f$,	$\frac{16}{2}f$,	etc.

Therefore the spaces described in the first, second, third, etc., seconds will be:

1st sec.	2d.	3d.	4th.	
$\frac{1}{2}f$,	$\frac{3}{2}f$,	$\frac{5}{2}f$,	$\frac{7}{2}f$,	etc.

In other words, the spaces described in the successive seconds are proportional to the numbers 1, 3, 5, 7, 9, etc.; for the n^{th} second the space will be by this law $\frac{2n-1}{2}f$, or for a falling body $\frac{2n-1}{2}g$; this is equal to the space passed over in $(n-1)$ seconds $[=\frac{1}{2}f(n-1)^2]$ subtracted from the space passed over in n seconds $(=\frac{1}{2}fn^2)$.

EXAMPLES.

II. *Uniformly Accelerated Motion* (Articles 22–28). *A. Falling Bodies* (*take* $g = 32$).

[It is to be understood in each case that the body falls from rest, and that the resistance of the air is neglected. It is to be remembered, also, that the assumption that the value of g is constant for points above the surface of the earth is not rigidly true.]

1. A body falls 15 seconds: Required (a) the velocity acquired;

(*b*) the whole distance fallen through; (*c*) the space passed over in the last second of its fall; (*d*) the space in the last three seconds.

2. A body has fallen through 5184 feet: Required (*a*) the time of falling; (*b*) the final velocity.

3. A body has acquired in falling a velocity of 512 feet per second: Required (*a*) the time of falling; (*b*) the distance fallen through.

4. A body in falling passed over 336 feet in the last second: Required (*a*) the time of falling; (*b*) the distance fallen.

5. A body in falling passed over 1008 feet in the last three seconds: Required (*a*) the time of falling; (*b*) the distance fallen through.

6. What is the ratio of the velocities of a falling body at the end of the first $\frac{1}{4}$, $\frac{1}{2}$, 1, 3, and $4\frac{1}{2}$ seconds? Find the actual velocities in this way, from the velocity at the end of 1 second (g).

7. What is the ratio of the spaces passed over by a falling body in $\frac{1}{4}$, $\frac{1}{2}$, 1, 3, $4\frac{1}{2}$ seconds? Obtain the respective distances in this way, from that of 1 second (16 feet).

8. A sand-bag is dropped from a balloon, which is for the moment at rest at a height of 3 miles: Required (*a*) the time of falling to the earth, and (*b*) the velocity acquired.

9. What is the distance fallen through in the third of a second, commencing (*a*) the 6th second, and (*b*) the 11th second?

10. Two balls are dropped at the same instant from points 100 feet apart vertically: What distance will separate them at the end of 2, 3, and 5 seconds?

11. Two balls A and B are dropped from a height, B 2 seconds after the other: (*a*) How far apart will they be after B has fallen 2, 3, and 5 seconds? (*b*) When will they be 416 feet apart?

12. A stone is dropped down a well 224 feet deep: How soon will the splash in the water be heard at the top if the velocity of the sound is 1120 feet per second (corresponding to a temperature of the air of about 60° F.)?

13. A stone is dropped from the top of a cliff, and after $6\frac{1}{2}$ seconds it is heard to strike the ground below: How high is the cliff, taking the velocity of sound as 1152 feet per second (temperature of air about 90° F.)?

B. General Case.—Acceleration = f.

[The motion is assumed to be uniformly accelerated.]

1. A body moves 100 feet in the first 5 seconds from rest What is the acceleration?

2. A body moves 10 feet in the first second: (*a*) What is the acceleration? (*b*) How far will it go in 8 seconds? (*c*) What will be its final velocity at the end of this time?

3. The acceleration is 12 feet-per-second per second: (*a*) What velocity does a body acquire in 6 seconds? (*b*) What space does it pass over?

4. A body passes over 36 feet in the fifth second: What is the acceleration?

5. The acceleration due to the attraction of Jupiter for bodies on or near its surface is about 2.6 times g: (*a*) What velocity would a falling body acquire in 3 seconds? (*b*) What space would it pass through in this time?

6. What time would be required in the above case (5) for a body to fall 2340 feet?

7. The acceleration of gravity on the moon is about $\frac{1}{6}g$: How long and how far must a body fall to acquire a velocity of 32 feet?

8. The acceleration of gravity on the sun is about $28 \times g$: Compare the acquired velocities and spaces fallen through for the first three seconds with those true for the earth.

9. A body moves 45 feet in 3 seconds, and 80 feet in the next 2 seconds: Is its motion uniformly accelerated?

10. A body passes over 50 feet in 5 seconds: What distance must it go in the next 5 to satisfy the condition of uniformly accelerated motion?

Composition and Resolution of Velocities—Uniform Motion.

29. Composition of Motions in General. It was explained in Art. 11 that, when a body is said to be in motion, reference is always made to some other body with respect to which the first body changes its posi-

tion. In many cases which arise we have to consider not the simple motions of bodies, but their actual motions as composed of several different motions. For example, a man walking on the deck of a steamboat is in motion with reference to it, but the boat in turn is in motion as compared with the neighboring shore. Therefore his actual motion with reference to the land is composed of his own independent motion and that of the boat. Hence we have, in such cases, to do with the *coexistence of motions;* and the problem arises, when the separate motions are given, to find the actual resulting motion in rate (velocity) and direction.

30. Resultant and Component Velocities. When a body tends to move at the same time with several different velocities, either in the same or different directions, the actual velocity due to the combination of all is called the *resultant,* and the separate velocities are called the *components.* The process of finding the resultant, when the components are given, is called the *Composition of Velocities.*

31. Composition of Constant Velocities in the same Straight Line. The resultant of two component velocities in the same direction is equal to their sum; if they have opposite directions, it is equal to their difference. In general, if of several velocities those in one direction are called plus (+), and those in the opposite are called minus (−), the resultant is equal to their algebraic sum.

For example, a boat, moving uniformly at the rate of 6 miles per hour down a stream running at the uniform rate of 4 miles, has a resultant velocity in the same direction of 10 miles $(6 + 4)$. If the boat is headed up stream and keeps the same rate, the resultant velocity is

also up stream and equal to 2 miles $(6-4)$. If, in the latter case, the stream had a velocity of 8 miles, the resultant velocity would be equal to $-2, (6-8)$; that is, the boat would in fact drift down stream at this rate.

32. Composition of two Constant Velocities not in the same Straight Line. If the two component velocities are not in the same straight line, then the resultant velocity will lie between them, and will be determined in direction and magnitude by the Parallelogram of Velocities.

33. Parallelogram of Velocities. This principle is stated as follows: *If the component velocities be represented in direction and magnitude by the two adjacent sides of a parallelogram, the resultant velocity will be given by the diagonal passing through their point of intersection.* Suppose a body tends to move uniformly

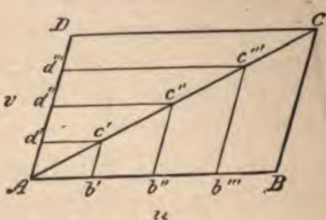


FIG. 8.

from A toward B (Fig. 8) with a velocity u, represented by the line AB; also at the same instant from A to D with a velocity v, represented by AD, then the body will actually move in the direction AC with a constant velocity represented by AC. (See Arts. 68, *b*, and 128.)

For the body, if it had only the velocity u, would in one second move from A to B, and if only the velocity v, would move from A to D; but the motion in the one

direction cannot effect that in the other if they go on together, so that at the end of the given time the body will actually be at C, having moved along the straight line AC. But, again, if Ab', $b'b''$, $b''b'''$, etc., be taken to represent the motion of the body in equal intervals of time in the direction AB with the velocity u, and Ad', $d'd''$, $d''d'''$, etc., the motion in the same intervals of time in the direction AD with the velocity v, the resultant motion will be represented by Ac', $c'c''$, $c''c'''$, etc. But these distances are equal, since they are by similar triangles proportional to the equal distances Ab', $b'b''$, etc. (or Ad', $d'd''$, etc.); therefore the motion of the body in the resultant direction is also uniform.

34. Calculation of the Magnitude and Direction of the Resultant Velocity. From trigonometry we have (see also Art. 130)

$$AC^2 = AB^2 + AD^2 + 2AB.AD \cos \gamma.$$

Therefore

$$V^2 = u^2 + v^2 + 2uv \cos \gamma.$$

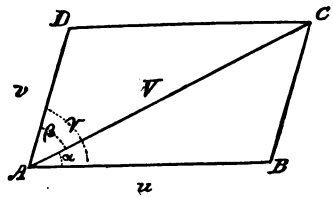


FIG. 9.

Also, in the triangle ABC (Fig. 9) the component velocities are represented by $AB = u$, and $BC (= AD) = v$; the resultant velocity is $AC = V$; also, $BAC = \alpha$, $ACB (= CAD) = \beta$, and $CBA = 180° - BAD = 180° - \gamma = 180° - (\alpha + \beta)$. Hence the relations in direc-

tion and magnitude of the resultant and component velocities may be calculated by the usual methods for the solution of this plane triangle, where three parts are given and the others required.

It is further seen from this case that the relation of the two component velocities and their resultant may be expressed geometrically by the triangle ABC, hence sometimes called the *Triangle of Velocities.*

35. For the special case (Fig. 10) where the directions

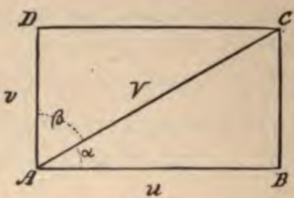


Fig. 10.

of the component velocities are at right angles to each other, the relations are more simple. Here

$$V = \sqrt{v^2 + u^2}; \text{ also, } \cos\alpha = \sin\beta = \frac{u}{V},$$

$$\sin\alpha = \cos\beta = \frac{v}{V},$$

$$\tan\alpha = \cot\beta = \frac{v}{u}.$$

36. The following cases may be taken as illustrations of the Parallelogram of Velocities.—Suppose a boat to move uniformly at the rate of u miles per hour across a stream running at the rate of v miles per hour. Here the velocities u and v are the components, and, if Ab, Ad be taken to represent them in their respective directions (Fig. 11), the resultant velocity will be given

in direction and magnitude by the diagonal Ac. Hence the boat will actually move in the direction Ac, $\left(\tan\alpha = \frac{v}{u}\right)$ at the rate of V miles an hour $(= \sqrt{v^2 + u^2})$, and will reach C in the same time in which it would have gone without the current directly across to B, or would have drifted with the current to D.

Again, suppose the component velocities as above; if it be required that the boat shall go directly across to C, here the direction of the resultant is AC (Fig. 12), and the components are

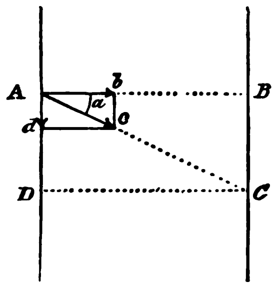


Fig. 11.

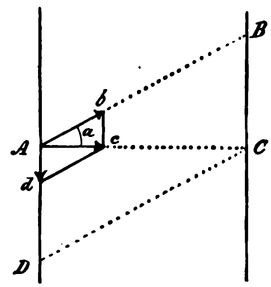


Fig. 12.

$Ab\,(= u)$ and $Ad\,(= v)$. Hence the boat must be headed up stream at an angle BAC, $\left(\sin\alpha = \frac{v}{u}\right)$, and the resultant velocity will be expressed by $\sqrt{u^2 - v^2}$.

37. Composition of several Constant Velocities. If there are more than two component velocities, then the method of finding their resultant is as follows: Find the resultant of two of the component velocities, then of this resultant and the third component, again of the last resultant and the fourth component, and so on. Thus let AB, AC, AD, AE (Fig. 13), represent several component velocities; the resultant of AB and AC is (33) the diagonal Ac; again, the resultant of Ac and AD, that

is of AB, AC, AD, is Ad; still again, the resultant of Ad and AE, that is of the four original velocities, is Ae.

It will be seen by comparing Figs. 13 and 14, that the sides of the polygon $a\ b\ c\ d\ e$ (Fig. 14) represent the four velocities and their resultant. Hence, in general, if the component velocities be laid off in order of direction, as ab, bc, cd, de (Fig. 14), the side which com-

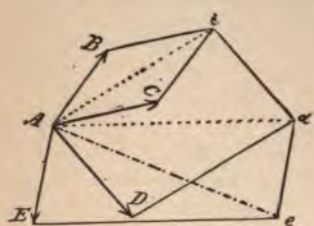


FIG. 13.

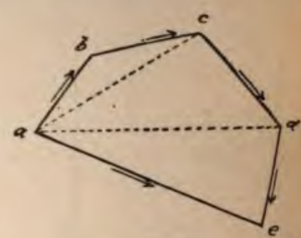


FIG. 14.

pletes the polygon so formed, viz. ae, represents the resultant velocity. This is sometimes called the *Polygon of Velocities.*

38. Resolution of Velocities. The process of finding the component velocities, which shall be equivalent to a given resultant velocity, is called the *Resolution of Velocities.* If the required components are two in number and their directions are given, then their magnitude is found by completing the parallelogram whose diagonal is the given resultant velocity and whose sides have the given directions.

Let (Fig 15) AC represent the resultant velocity, and let AX and AY be the directions of the required components; draw through C the lines CD, CB parallel respectively to AX and AY, then AB and AD, the

sides of the parallelogram thus formed, will represent the required components in direction and magnitude.

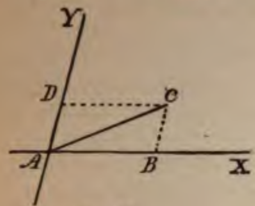


FIG. 15.

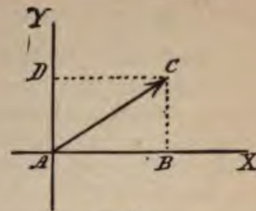


FIG. 16.

If the directions AX and AY are at right angles (Fig. 16), and α represents the angle CAB, then the components are $AB = AC \cos \alpha$ and $AD = AC \sin \alpha$.

For example, suppose a boat to move uniformly in the direction AC (Fig. 17) in virtue of its own motion

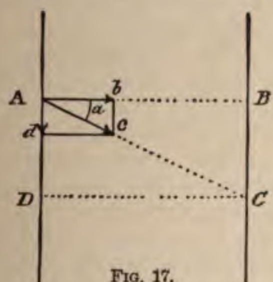


FIG. 17.

directly across the stream and the velocity of the current toward D, taken together, both being uniform. Then if Ac represents this resultant velocity, the component velocities will be given by Ab $(= Ac \cos \alpha)$ and Ad $(= Ac \sin \alpha)$. If the transfer across the stream were alone desired, the component Ab in this direction might be called the *effective velocity*.

EXAMPLES.

III. *Composition of Velocities.* Articles 29–37.

[The velocities are supposed to be constant in all cases.]

1. The velocity of a steamboat is 5 miles per hour, that of the stream is 4 miles, and a man walks the deck from stern to bow at the rate of 3 miles: Required the actual velocity of the boat (*a*) if headed up stream, and (*b*) down stream; also (*c*, *d*), that of the man in each case.

2. The velocities of boat and stream are as in example 1, and the boat is headed directly across the stream (Fig. 11): (*a*) What will be the actual direction of the boat's motion? (*b*) What the rate of its motion? (*c*) How long will the passage take if the stream is 2 miles wide? (*d*) Where will the boat land ($BC = ?$).

3. The velocities of boat and stream are as in 1 and 2, but it is required that the boat shall go directly across from A to C (Fig. 12): (*a*) In what direction must the boat be headed? (*b*) What will be its actual velocity across? (*c*) What will be the time of passage, the width being 2 miles?

4. Find answers for the three questions in example 3, on the supposition that the boat is to reach a point 30° up stream from the starting-point.

5. Find answers for the three questions in example 3, on the supposition that the boat must reach a point 30° down stream.

6. The velocity of the boat is 4 miles per hour, and that of the stream 5 miles; the width of the stream is 1 mile: What is nearest point, to that directly across from the starting-point, which the boat can reach? (Solve this problem by geometrical construction.)

7. A ball on a horizontal surface tends to move north with a velocity of 12 feet per second, and east with a velocity of 5 feet per second: (*a*) What will be the actual velocity, and (*b*) in what direction?

8. A ball, moving north at a rate of 8 feet per second, receives an impulse tending to make it move due north-east with the same velocity: (*a*) What path will it take, and (*b*) at what rate will it move?

9. A man, skating uniformly at a rate of 12 feet per second,

projects a ball on the ice in a direction at right angles to his motion at a rate of 9 feet per second: What is (*a*) the actual rate, and (*b*) the direction of its motion (friction neglected)?

10. A ball tends to move north at the rate of 8 feet per second, also S. 60° E. and S. 60° W., each at the same rate: What is its actual velocity?

11. A ball tends to move east 5 miles per hour, also N. 45° W. and S. 45° W., each at the same rate: Required the direction and rate of motion.

12. If a boat headed directly across a stream moves at the uniform rate of 100 yards a minute, while the current runs 80 yards a minute: (*a*) In what direction will it actually go, and (*b*) what distance will it land down stream? (*c*) What should be its course in order that it may reach a landing-place directly opposite the starting-point, and (*d*) how long would the passage take? The width of the stream is 1200 yards.

IV. *Resolution of Constant Velocities.* Article 38.

1. A ball tends to move in a certain direction at a rate of 9 feet per second, but it is constrained to move at an angle of 30° with this direction: Required its velocity in the latter direction. (Fig. 16.)

2. A body moves uniformly about a semi-circumference at the rate of 12 feet per second: What is the component of its velocity parallel to the diameter when it is 30°, 60°, 90°, 120°, and 180° from the starting-point? (Fig. 16)

3. A ball rolls at the rate of 8 feet per second across the diagonal of a rectangular room $ABCD$ whose dimensions are 15×20 $(= AB \times AC)$: What is its rate of motion parallel to each side?

4. A body moves N. 30° E. at a rate of 6 miles per hour: Required its rate of motion northerly and easterly?

5. A boat, though headed directly across a stream, actually moves diagonally across the stream at an angle of 30° (BAC, Fig. 11, p. 27), and at a rate of 10 miles per hour: Required (*a*) the rate of the boat, and (*b*) of the current, each taken independently.

6. A boat steams directly across a stream 1800 yards wide in 30 minutes, the current flowing all the time at the rate of 80 yards per minute: What would be the direction and rate of motion if there were no current?

7. A balloon has a velocity of 20 feet per second in an upward direction which makes an angle α with a vertical line: If its velocity vertically upward would be 1000 feet per minute, what is its horizontal velocity due to the wind? What is α?

Composition and Resolution of Accelerations.

39. Composition and Resolution of Accelerations. The composition and resolution of velocities may be extended also to the case of uniform accelerations, the method being in all respects similar to that in the preceding articles. The sides of the parallelogram here represent the component accelerations, and the diagonal the resultant acceleration.

The simplest application of the principle of the resolution of accelerations is to the case of motion down an inclined plane (40).

40. Motion down an Inclined Plane. The direction of the acceleration of gravity is that of a vertical line, and

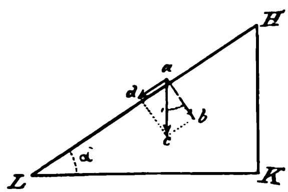


FIG. 18.

a body falls in this direction if entirely free; but a body on an inclined plane is only free to slide along it, and the acceleration is here that component of the whole acceleration which is parallel to the plane: viz., $g \sin \alpha$.

Let (Fig. 18) ac be taken to represent the vertical acceleration g; the directions of its components are respectively parallel and perpendicular to the plane, and

are represented by ab and ad. But $bac = HLK = \alpha$, and therefore $ad = bc = ac \sin \alpha$. That is, ad, or the acceleration down the plane, is equal to $g \sin \alpha$.

The formulas of Art. 27, for a falling body, are then applicable to the case of a body sliding down a smooth inclined plane, if for g we write $g \sin \alpha$. That is:

$$v = g \sin \alpha \, . \, t,$$
$$s = \tfrac{1}{2} g \sin \alpha \, . \, t^2,$$
$$v^2 = 2g \sin \alpha \, . \, s.$$

In the last formula, if the body descends from H to L, $s = HL$, and $s \sin \alpha = HK$ or h, the height of the plane;

$$\therefore v^2 = 2gh.$$

From this equation it follows that: *the velocity acquired in descending any inclined plane is the same as that gained in falling through the vertical height of the plane.* This is also true for a continuous curve. This principle finds an application in Art. 244.

EXAMPLES.

V. *Falling down an Inclined Plane.* Article 40.

[The plane is supposed to be perfectly smooth, so that there is no friction.]

1. The angle of the plane is 30°: Required (a) the acceleration down the plane; (b) the distance fallen through in 4 seconds; (c) the velocity acquired; (d) the distance in the last second.

2. The height of the plane is 100 feet and the length 400: (a) What is the time required to reach the bottom? (b) What is the velocity acquired?

3. The angle of the plane is 45°: Required the time of falling 144 feet.

4. The length of a plane is 576 feet, a body falls down it in 24 seconds: (*a*) What is the acceleration? (*b*) What is the height of the plane?

5. The height of a plane is 98 feet, and a body gains a velocity of 20 feet per second in falling 5 seconds on it: Required (*a*) the acceleration; (*b*) the length of the plane.

6. The height of a plane is 256 feet, a body reaches the bottom in 16 seconds: (*a*) What is the length of the plane? (*b*) What is the velocity acquired?

7. Several planes, having the same altitude, viz. 400 feet, have lengths 600, 800, 1200, and 1600 feet: Compare the times of descent and acquired velocities for each.

8. Show that for several planes having the same altitude the times of descent are proportional to the lengths; that is, $\frac{t}{t'} = \frac{l}{l'}$ or $t \propto l$.

9. Prove that the time of falling from rest down a chord of a vertical circle, drawn from the highest point, is constant.

Composition of Uniform and Accelerated Motion in the same Line.

41. Composition of Uniform and Accelerated Motion in the same Straight Line. (Only the cases of uniformly accelerated and retarded motion will be considered.)

If of two component velocities one is constant (u) and the other is uniformly increasing—that is, tending to produce uniformly accelerated motion—but both in the same line, then the resultant is equal to their sum or difference according as they have the same (*a*) or opposite directions (*b*).

(*a*) In the first case, represent the uniform velocity by u, and that produced by the accelerated motion by v $(= ft)$; then, if V is the resultant velocity,

$V = u + v = u + ft$; for a falling body $V = u + gt$. (1)

The last formula applies to the case of a body projected with an initial velocity vertically downward toward the surface of the earth from a point above. The resultant velocity is the sum of this initial velocity and that due to its accelerated motion caused by gravity (24).

(*b*) In the second case

$$V = u - v = u - ft; \text{ for a falling body } V = u - gt. \quad (2)$$

The last formula here applies to the case of a body projected vertically upward from the earth with an initial velocity u; its resultant velocity at any moment is then equal to the initial velocity diminished by the velocity due to the accelerated motion downward (that is, in the opposite direction) caused by gravity.

42. The distance (s) which a body passes over in a given time, in the above examples, is to be found by taking the sum, in the first case, and the difference, in the second, of the space that would be passed over if it moved uniformly for the time t (19) with the velocity u, and that it would pass over independently in the same time in consequence of the accelerated motion (27).

Therefore (*a*)

$$s = ut + \tfrac{1}{2}ft^2; \text{ for a falling body } s = ut + \tfrac{1}{2}gt^2. \quad (3)$$

And (*b*)

$$s = ut - \tfrac{1}{2}ft^2; \text{ for a falling body } s = ut - \tfrac{1}{2}gt^2. \quad (4)$$

By combining equations (1) and (3), since $V = u + ft$,

$$V^2 = u^2 + 2uft + f^2t^2 = u^2 + 2f(ut + \tfrac{1}{2}ft^2) = u^2 + 2fs.$$

Therefore

$$V^2 = u^2 + 2fs; \text{ for a falling body } V^2 = u^2 + 2gs. \quad (5)$$

In the same manner

$$V^2 = u^2 - 2fs; \text{ for a falling body } V^2 = u^2 - 2gs. \quad (6)$$

43. Geometrical Representation. It was shown, in Art. 21, that the space passed over by a body moving uniformly may be represented geometrically by a rectangle; and again, in Art. 25, that the space described by a body moving with uniformly accelerated motion may be represented by a right-angled triangle. If now a body has an initial velocity in the same or opposite direction to that in which it begins to move with uniformly accelerated motion, the space passed over will be represented by a geometrical figure formed by the combination of the rectangle and triangle.

For example, in Fig. 19, let AB represent the time (t), BC the initial velocity (u), also CE the velocity (v) acquired in this time and in the same direction as u; then will the whole space passed over be represented by the figure $ABED$, which is the sum of the rectangle $ABCD$ (ut) and the triangle DCE ($\frac{1}{2}vt = \frac{1}{2}ft^2$).

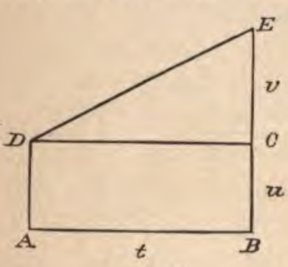


Fig. 19.

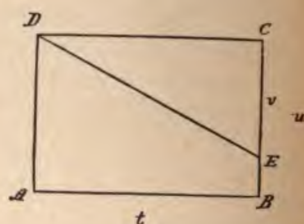


Fig. 20.

Again, suppose the accelerated motion to be in a direction opposite to that of the initial velocity. Let AB (Fig. 20) represent the time (t), and BC the initial velocity (u), also take CE to represent the velocity acquired (v) in the given time; then the space described will be proportional to the area of the quadrilateral

ABED, which is the difference between the rectangle *ABCD* (ut) and the triangle *DCE* ($\frac{1}{2}vt = \frac{1}{2}ft^2$).

44. Motion of a Body projected vertically upward. The three formulas obtained in Art. 42, which give the relations of the *velocity, space,* and *time* of a body which has an initial velocity in a direction *opposite* to that in which it tends to move with accelerated motion, have an especial importance. They are:

$$V = u - ft, \qquad s = ut - \tfrac{1}{2}ft^2, \qquad V^2 = u^2 - 2fs.$$

For a falling body these are:

$$V = u - gt, \ (1) \quad s = ut - \tfrac{1}{2}gt^2, \ (2) \quad V^2 = u^2 - 2gs. \ (3)$$

The relations given below are deduced from the last three equations, since the case of the body projected vertically upward is practically the most important, but all the results obtained may be made general by writing f for g.

1. *The Time of Ascent.* From equation (1), if $t = 0$ —that is, at the moment of starting—$V = u$, the initial velocity; as t increases V diminishes, and when $gt = u$, or $t = \frac{u}{g}$, then $V = 0$. That is, at a time after the starting, expressed by $t = \frac{u}{g}$, the body will for an instant come to rest.

2. *Time of Descent.* If in the same equation gt is greater than u, i.e. t is greater than $\frac{u}{g}$, the value of V will be negative; in other words, the body will begin to descend. When it reaches the starting-point again, $s = 0$, and therefore, from equation (2),

$$ut - \tfrac{1}{2}gt^2 = 0, \qquad \text{and} \qquad t = 0 \text{ or } \frac{2u}{g}.$$

The value $t = 0$ corresponds obviously to the moment of starting, and $t = \frac{2u}{g}$ means that at the end of this time the body will have returned to the starting-point. The time of ascent and descent is then $\frac{2u}{g}$; and since the former $= \frac{u}{g}$, the time of descending must be also equal to $\frac{u}{g}$.

3. *Height of Ascent.* At the highest point reached $V = 0$, and therefore in equation (3)

$$0 = u^2 - 2gs, \qquad \text{and} \qquad s = \frac{u^2}{2g},$$

and this value gives the distance ascended.

This can also be obtained from equation (2); for at the highest point $t = \frac{u}{g}$, therefore

$$s = \frac{u^2}{g} - \frac{u^2}{2g} = \frac{u^2}{2g}.$$

This equation is the same as (3) in Art. 27; hence the result here obtained may be stated in this form: A body projected vertically up will ascend to a height from which it must fall to acquire a velocity equal to that of its projection.

4. *Velocity acquired in descending.* The time required for the whole ascent and descent is $\frac{2u}{g}$, therefore in equation (1)

$$V = u - g \cdot \frac{2u}{g} = u - 2u = -u,$$

or the velocity acquired in descending is equal to the initial velocity, but in the opposite direction.

If in equation (3) we let $s = 0$, then

$$V^2 = u^2, \qquad V = +u \text{ or } -u,$$

which result corresponds to that just given.

Furthermore, for any value of s there will be two different values of t from equation (2), corresponding to the time when it passes the given point on the ascent, and that when it returns to it on the descent. Also, at any point on the descent the velocity (equation 3) will be the same with the contrary sign as that on the corresponding point in the ascent.

45. Projected up or down an Inclined Plane. For a body moving up or down a smooth inclined plane with an initial velocity u, the relations are the same as those given in articles 41, 42, and 44, except that, as in 40, in every case we must write $g \sin \alpha$ for g.

Down.	Up.
$V = u + g \sin \alpha . t,$	$V = u - g \sin \alpha . t,$
$s = ut + \frac{1}{2} g \sin \alpha . t^2,$	$s = ut - \frac{1}{2} g \sin \alpha . t^2,$
$V^2 = u^2 + 2g \sin \alpha . s.$	$V^2 = u^2 - 2g \sin \alpha . s.$

EXAMPLES.

VI. *Bodies projected vertically downward.* Articles 41, 42.

[The resistance of the air is neglected.]

1. A body is thrown vertically down with an initial velocity of 36 feet per second: Required (*a*) the velocity at the end of 7 seconds; (*b*) the distance fallen through; (*c*) the space passed over in the last second.

2. A body is projected down with an initial velocity of 20 feet per second: (*a*) How long will it require to fall 594 feet? (*b*) What velocity will it then have?

3. What velocity of projection must a stone have to reach the bottom of a cliff 270 feet high in 3 seconds ?

4. With what velocity must a stone be thrown down the shaft of a mine 556 feet deep in order that the sound of its fall may be heard at the top after $4\frac{1}{4}$ seconds ? The velocity of sound is to be taken as 1112 feet per second.

5. A body projected vertically down has a velocity of 215 feet per second at the end of 5 seconds: Required (*a*) the velocity of projection; (*b*) the distance gone through.

6. A body projected down passes over 133 feet in the fourth second: Required the velocity of projection.

7. A stone is dropped from a bucket which is descending a shaft at the uniform rate of 12 feet per second, and at the moment when the bucket is 238 feet from the bottom: (*a*) How far will they be apart in 2 seconds ? (*b*) When will the stone reach the bottom ?

8. A sand-bag is dropped from a balloon which is descending at the uniform rate of 24 feet per second; after 8 seconds it strikes the ground: (*a*) What was the height of the balloon ? (*b*) How far were they apart after 5 seconds ?

VII. *Bodies projected vertically upward.* Article 44.

1. The velocity of the projection upward is 288 feet: Required (*a*) the time of ascent; (*b*) of descent; (*c*) the height of ascent; (*d*) the distance gone in the first and last seconds of ascent.

2. A body is projected up with a velocity of 192 feet per second: (*a*) When will it be 432 feet above the starting-point ? (*b*) When will it be 720 feet below the starting-point ? Explain the double answer in each case.

3. A body is projected up with a velocity of 208 feet: How long after starting will its velocity be (*a*) + 64, also (*b*) − 64 and (*c*) − 272 ? (The minus sign indicates downward motion.)

4. What velocity of projection must a ball have in order to ascend just 900 feet ?

5. What time does a body require to ascend 2304 feet, that being the highest point reached ?

6. What velocity of projection is needed to make a body ascend just 6 seconds ?

7. A ball thrown up passes a staging 96 feet from the ground at the end of 1 second: (*a*) What was the velocity of projection? If the time is 6 seconds (*b*), what is the answer?

8. A body projected up passes over 112 feet in the fifth second of its ascent: What was its velocity of projection?

9. A and B are two points 40 feet apart in a vertical line; a ball is dropped from A, and at the same instant one thrown up from B with a velocity $= 80$ feet per second: When and where will they pass each other?

10. A ball is dropped from the top of a cliff, and at the same instant another is thrown up with a velocity of 176 feet per second: (*a*) If they pass each other at the end of $2\frac{1}{2}$ seconds, how high is the cliff? (*b*) How far were they apart at the end of 2 seconds? (*c*) at the end of 3 seconds?

11. A ball is thrown up from the ground with a velocity of 128 feet per second, and 2 seconds later another is thrown with a velocity of 160 feet: When and where will they pass each other?

12. A bucket is ascending a shaft uniformly at a rate of 32 feet per second: What will be the apparent motion of a stone dropped from it (*a*) to a person in the bucket? (*b*) to a person on the side of the shaft opposite the initial point?

13. A balloon is rising uniformly at the rate of 96 feet per second; at the instant it is 640 feet from the ground a sand-bag is dropped from the car: What will be the motion of the bag, when will it reach the ground, and over what distance will it have passed?

VIII. *Projected up or down a smooth Inclined Plane.* Article 45.

1. The height of the plane is 114 feet, the length is 456, the velocity of projection down is 25 feet per second: (*a*) How long will it require to descend? (*b*) What will be the final velocity?

2. The height and length are 144 and 576 feet respectively: (*a*) What velocity of projection up is required that it may just reach the top? (*b*) What time will it take?

3. The angle of the plane is 30°, the velocity of projection down is 45 feet: Required (*a*) the velocity at the end of 4 seconds; (*b*) the distance gone through; (*c*) the distance in the last second.

4. The angle of the plane is 30°, the velocity of projection up is

80 feet per second: Required (*a*) the length of time the body will continue to go up, (*b*) the distance gone, and (*c*) the velocity at the end of 2 and of 8 seconds.

IX. *Bodies projected against Friction.* Articles 41, 42.

[The retardation (or minus acceleration) due to friction takes the place of the *f* in the formulas of articles 42 and 44.]

1. A body projected on a *rough* horizontal plane has at starting a velocity of 120 feet per second, but loses this at the rate of 12 feet for each succeeding second: (*a*) What is the retardation (minus acceleration) due to friction? (*b*) When will the body stop? (*c*) How far will it have gone?

2. The retardation due to friction is for each second 8 feet per second for a given sliding body, the initial velocity is 40 feet per second: Required (*a*) the time it will continue to slide; (*b*) the distance it will go; (*c*) its velocity at the end of 3 seconds.

3. A railroad-car, when the engine is detached, has a velocity of 15 miles per hour, the retardation due to friction is 1 foot-per-second per second: How far and how long will the car continue to move?

4. If the retardation of friction is 4 feet-per-second per second: (*a*) What initial velocity (in miles per hour) must a body have in order to slide just 968 feet? (*b*) If the velocity is doubled, how much farther will it go?

5. A body is projected up a rough inclined plane at an inclination of 30°, the retardation of friction alone is 4 feet-per-second per second: If the initial velocity is 400 feet per second, how far and for how long will the body ascend?

Composition of Uniform and Accelerated Motion not in the same Straight Line.

46. Composition of Uniform and Accelerated Motion. If a body tend to move in one direction with uniform motion, and in another direction with accelerated motion, its actual path is not a straight line, but a *curve.* The same principle involved in the Parallelogram of

Velocities (33) makes it possible to determine the position of the body at the end of any given time (read Art. 68, *b*, p. 68). For if one side of the parallelogram represents, in direction and amount, the uniform motion in the given time, and the adjacent side the corresponding accelerated motion, the diagonally opposite point of the parallelogram will indicate the actual position of the body at the end of this time. In this statement nothing is said about the path which the body has described.

47. Projectile. The simplest application of the above principle is to the case of the projectile. It will be

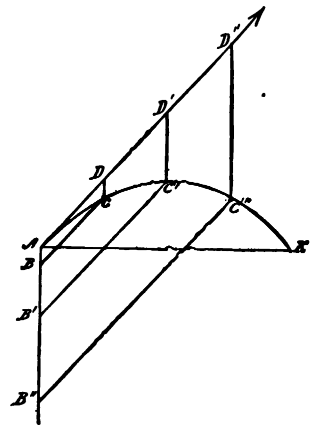


FIG. 21.

shown that, if the resistance of the air be neglected, *the path of a projectile is a parabola.*

Suppose a body starts from A in the direction AD (Fig. 21) with an initial velocity u; at the end of t seconds, if no other motion were imparted to it, it would reach a point D, so that

$$AD = ut. \qquad (1)$$

But from the instant of starting it falls vertically downward under the influence of gravity, with uniformly accelerated motion. At the end of t seconds, if it had no initial velocity, it would fall to B, so that

$$AB = \tfrac{1}{2}gt^2. \qquad (2)$$

But as shown above (46), as the body must obey both tendencies to motion simultaneously, its actual position at the given time will be at C.

Squaring (1) and dividing by (2), we have

$$\frac{AD^2}{AB} = \frac{BC^2}{AB} = \frac{u^2t^2}{\frac{1}{2}gt^2} = \frac{2u^2}{g},$$

or

$$\frac{BC^2}{AB} = \frac{2u^2}{g}.$$

Therefore the ratio of the square of the ordinate BC ($B'C'$, $B''C''$) to the abscissa AB (AB', AB'') is constant, and hence the curve is a parabola.

48. Position of the Directrix, Axis, Focus. The line AD (Fig. 22) is a tangent to the parabola at A, the verti-

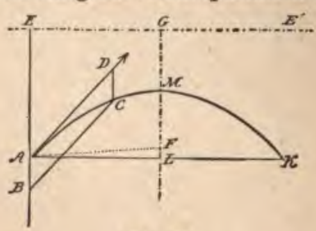


FIG. 22.

cal line EAB is a diameter. The constant value of the ratio of $\frac{BC^2}{AB}\left(=\frac{2u^2}{g}\right)$ is four times the distance from A to the directrix or to the focus. The same relation is at

once obvious in the case of the parabola, whose equation is $y^2 = 4ax \left(\text{i.e. } \frac{y^2}{x} = 4a\right)$; it may also be proved analytically for this case, where the co-ordinates EB, AD are oblique. It is proved geometrically in a following paragraph (50, *e*).

If AE is taken on the vertical line equal to $\frac{u^2}{2g}$, and EGE' be drawn horizontally, this line will be the directrix; and if from A on the line AF (drawn so that the angle $EAD = DAF$) we take $AF = \frac{u^2}{2g}$, the point F is the focus of the parabola. The vertical line GML is the axis.

If the direction of the initial velocity be horizontal, as in Fig. 23, then the starting-point A is the vertex, the vertical line AB is the axis, and the focus and directrix are at distances equal to $\frac{u^2}{2g}$ from A. This figure shows well, as does also Fig. 21, what is meant by the statement in (27), that in uniformly accelerated motion the space is proportional to the square of the time. Here, if the successive intervals of time are equal,

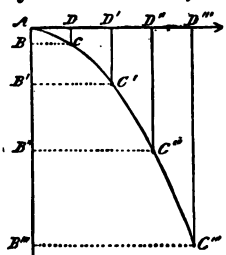


FIG. 23.

$$AD : AD' : AD'' : AD''', \text{ etc.}, = 1 : 2 : 3 : 4,$$

and

$$AB : AB' : AB'' : AB''', \text{ etc.}, = 1 : 4 : 9 : 16.$$

49. The *actual* path of a projectile deviates widely from a parabola because of the resistance of the air, which is very great with high velocities, as that of a cannon-ball or rifle-bullet (perhaps 1600 feet per second

in starting). For this reason the maximum distance is gained, not by an angle of 45° (as shown below), but for an angle of a little over 30°.

A jet of water illustrates the subject of the projectile well, since each particle may be considered as an independent projectile, and thus the shape of the jet gives the continuous path. It shows, moreover, the deviation caused by the resistance of the air.

50. Time of Flight, Range, etc. In Fig. 24 the angle

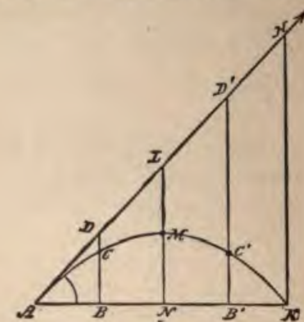


FIG. 24.

HAK is called the *angle of projection,* and the horizontal distance *AK* is the *range.*

(*a*) From the triangle AHK, $AH = ut$ and $HK = \frac{1}{2}gt^2$; also,

$$\sin\alpha = \frac{HK}{AH} = \frac{\frac{1}{2}gt^2}{ut} = \frac{gt}{2u},$$

$$\therefore\ t = \frac{2u\sin\alpha}{g}. \qquad (1)$$

The value of t in (1) gives the *time of flight.*

(*b*) Again,

$$AK = AH\cos\alpha = ut.\cos\alpha;$$

or, substituting the above value of t,

$$AK = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{g}. \qquad (2)$$

This value AK gives the *range*. Further, since $\sin 2\alpha = \sin (180° - 2\alpha) = \sin 2(90° - \alpha)$, it is obvious that for every horizontal distance there are two values of the angle of projection; viz., α and $(90° - \alpha)$. This is indicated in Fig. 25. The time of flight for the angle α is $\frac{2u \sin \alpha}{g}$, and for $(90° - \alpha)$ it is $\frac{2u \cos \alpha}{g}$.

The maximum range is obtained when $\alpha = 45°$ and

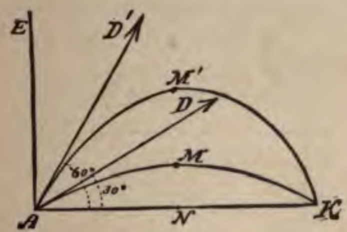


FIG. 25.

$\sin 2\alpha = 1$, for the value of AK is then the greatest; for this case the two paths of the projectile coincide.

(*c*) Since the vertical component of the initial velocity is $u \sin \alpha$, the actual vertical velocity of the projectile will be given, for any time t, by the formula (41)

$$V = u \sin \alpha - gt. \qquad (3)$$

For the highest point $V = 0$, and hence $t = \frac{u \sin \alpha}{g}$, and combining this with (1), it is seen that the times of ascent and descent are the same.

(*d*) The distance, CB (Fig. 24), of the projectile above

the horizontal line AK is given, for any time t, as follows:

$$CB = DB - DC = ut.\sin\alpha - \tfrac{1}{2}gt^2. \qquad (4)$$

For the highest point (M) $t = \dfrac{u\sin\alpha}{g}$, and hence

$$MN = \frac{u^2\sin^2\alpha}{g} - \frac{u^2\sin^2\alpha}{2g} = \frac{u^2\sin^2 a}{2g}. \qquad (5)$$

This value is greatest when $\sin\alpha = 1$ and $a = 90°$, in which case the parabola becomes a double straight line.

All the above results might have been obtained [as was (3) indeed] by the formulas in articles 41, 42, only taking $u\sin\alpha$ for u, s being the distance above or below $(-s)$ the horizontal plane.

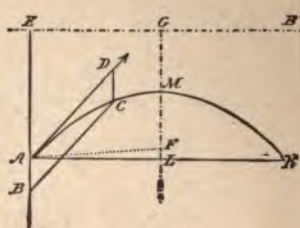


FIG. 26.

(e) We may prove geometrically the point mentioned in Art. 48; namely, that the distance from A to the focus is equal to $\dfrac{u^2}{2g}$; that is, to one fourth of the constant value of the ratio of the square of the ordinate to the abscissa $\left(\dfrac{BC^2}{AB} = \dfrac{2u^2}{g}\right)$.

Draw the line AF (Fig. 26) so that the angle $DAF = DAE$; then, by the properties of the parabola, the focus

must lie in this line; it must also lie on the axis GL, and hence will be at F, their point of intersection. Now

$$AF = \frac{AL}{\cos FAL} = \frac{AL}{\cos(90° - 2\alpha)} = \frac{AL}{\sin 2\alpha}.$$

From equation (2) above,

$$AL = \tfrac{1}{2}AK = \frac{u^2 \sin 2\alpha}{2g},$$

$$\therefore AF = \frac{u^2 \sin 2\alpha}{2g} \div \sin 2\alpha,$$

or

$$AF = \frac{u^2}{2g}.$$

The line EGE' is the common directrix of all the parabolas described by projectiles having the same initial velocity but different angles of projection. The foci of all these parabolas lie on the circumference of a circle having A as its centre and a radius equal to $\frac{u^2}{2g}$.

51. The theory of the projectile may be further illustrated by the case of a jet of water flowing from a lateral orifice in the vertical side of a reservoir (Fig. 27). By a principle of hydrostatics the initial velocity of flow is the same as that which would be gained in falling freely through the height from the top of water to the orifice (27); that is, $u = \sqrt{2g.AC}$, or $AC = \frac{u^2}{2g}$.

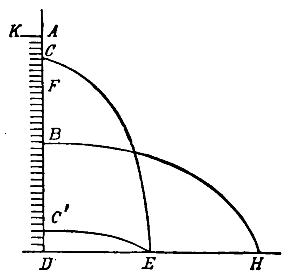


FIG. 27.

Supposing now that the level of the water is kept uniform, the directrix will coincide with it, that is, AK; the focus will be at F, so that $CA = CF = \frac{u^2}{2g}$. Further, it may be readily shown that the

range DE is the same for any two points taken, as C and C', so that $AC = C'D$, and finally that the maximum distance DH is gained by an aperture in the middle at B, and is equal to $AD(=2AB)$.

EXAMPLES.

X. *Projectiles.* Articles 47–51.

[The resistance of the air is left out of account.]

1. The initial velocity of a projectile is 160 feet per second, and the angle of elevation is 30°: Required (*a*) the time of flight; (*b*) the range: (*c*) the highest point reached.

2. When will the ball in example 1 be 96 feet above the ground? Explain the double answer.

3. The initial velocity is 320 feet per second: What angle of elevation will give a range of 800 feet? Show that there are two answers.

4. The angle of elevation is 15°: What initial velocity is required that the range should be .4 miles?

5. A rifle-ball is shot horizontally from the top of a tower 100 feet high, and with an initial velocity of 1200 feet per second: When and how far from the base of the tower will it strike the horizontal plane below?

6. A ball is thrown horizontally from the top of a cliff above the sea; it strikes the water in 5 seconds and at a horizontal distance of a mile: What was (*a*) the initial velocity, and (*b*) what was the height of the cliff?

7. If (Fig. 27) apertures are made at two points 36 feet from the top and bottom of the reservoir respectively, the whole height being 136 feet, what will be (*a*) the horizontal distance reached by the water in each case, and what (*b*) the initial velocity?

8. A stone is dropped from the top of a railroad-car, 16 feet above the ground, and when it is moving at the rate of 45 miles per hour. What will be its apparent motion (*a*) to a person on the train, (*b*) to one standing by the track? (*c*, *d*) When and where will it reach the ground?

9. At what angle of elevation must a projectile be fired in order that it may strike an object 2500 feet distant on the same horizontal plane, the velocity of projection being 400 feet per second?

CHAPTER II.—DYNAMICS.

52. The preceding chapter was devoted to the subject of KINEMATICS, or the discussion of the motion of bodies without reference to their mass or to the force or forces which cause the motion. These latter subjects, included under DYNAMICS, or KINETICS, are considered in the present chapter.

Mass—Density—Volume—Momentum.

53. Mass or Quantity of Matter. *The mass of a body is the quantity of matter it contains.*

The relation in mass or quantity of matter, of different bodies of the same substance, and of uniform density (as defined in 56), is obviously given by the ratio of their volumes. For example, the mass or quantity of matter in a hundred cubic feet of iron is ten times that in ten cubic feet. For bodies of uniform density then: *the mass is proportional to the volume.*

In general, however, for bodies of different substances it is possible to compare their masses only as the effect of a known force upon them is observed. Thus, we judge roughly as to whether a barrel is empty or full, and, in the latter case, as to the nature of the contents by noting the degree of resistance which it offers to a force tending to move it; *e.g.*, a push or a kick. Similarly, if a ball of wood and another of the same size, of lead, attached to strings of equal length, be whirled

around at the same rate, the pull of the lead upon the centre will be the greater, and we form a rough estimate as to the relation of mass in this way. Could the pull at the centre be exactly measured under precisely the same conditions in each case, by means of a spring, the result would give the true relation of mass.

Still, again, could the velocities given by the same force to two bodies in equal times be exactly determined, their ratio would give also the ratio of the masses of the bodies. No one of these methods of estimating the mass can be conveniently employed in practice.

54. Mass determined by Weight. The simplest and at the same time most accurate method of comparing the masses of two bodies is by their *weight*, for the weight, or measure of the earth's attraction upon them determined by the balance, is, as proved by various experiments, proportional to the mass. Take two bodies of the same material, as two lumps of lead: if the weight of the first is twice that of the other, then it is easy to see that its mass, or the quantity of matter it contains, is also twice as great. But this is true in general: of two lots of lead and cotton, the bulk or volume of the latter may be much greater than that of the other, but if they have the same weight they have also the same mass; and if the weight of the lead is ten times that of the cotton, its mass is also ten times greater.

Mass may be measured then by weight, and, in ordinary language, the latter word, expressed, for example, in pounds, is used as standing for the mass. In this sense, *the* UNIT OF WEIGHT, *the pound, may be taken as also the* UNIT OF MASS (see articles 71, 72).

But the weight is also used as a measure of the force of gravity and of other forces compared with it. Hence

it must be carefully noted here that the term weight is employed with two distinct meanings, which should not be confounded; namely—

(*a*) As a measure of the mass or quantity of matter in a given body.

(*b*) As a measure of force by reference to the force of gravity.

55. Distinction between Mass and Weight. Although the weight of a body may properly stand for its mass, if their true relation is understood, the two terms are not identical. The mass or quantity of matter of a lead ball is the same wherever it is situated on the earth's surface; but the weight, which may be registered on a spring-balance, is slightly greater at the poles than at the equator. Again, at the surface of the sun the force of attraction on the same piece of lead, registered as before by the stretching of a spring, would be about twenty-eight times greater than on the earth. Still further, if we conceive of it as at a point in space far away from attracting bodies, there would be no sensible pull on the spring, nothing to correspond to the terrestrial weight, but the *mass* would be everywhere the same.

56. Relation between Mass, Density, and Volume. The *density* of a body is the mass or quantity of matter in the unit of volume. In comparing different bodies the density of water at the temperature of 39.2° F. (4° C.) is generally taken as unity; the fact that the weight—that is, the mass—of a given volume of lead is $11\frac{1}{2}$ times, or of iron 7 times, that of the same volume of water is expressed by saying that the density of lead is $11\frac{1}{2}$, and of iron is 7. A body is said to be throughout of uniform density when equal volumes, however small, have the same mass.

In Art. 53 it was stated that for bodies of uniform density the mass is proportional to the volume. It also follows that for bodies of equal volume *the mass is proportional to the density.*

Therefore, in general, the mass (M) is proportional to the product of the volume (V) and density (D). This may be expressed mathematically in this form:

$$M \propto DV; \text{ that is, } \frac{M}{M'} = \frac{DV}{D'V'};$$

whence

$$D \propto \frac{M}{V}, \quad \text{and} \quad V \propto \frac{M}{D}.$$

57. Momentum. *The momentum of a body is equal to the product of the mass and velocity.* Two bodies of the same mass, and moving with the same velocity, have obviously the same momentum. If these two bodies were joined together, still retaining the same velocity as before, the momentum of the two together as a whole would be twice that of either of them separately. In general, of two bodies having the same velocity, if the mass of one is five times that of the other, its momentum will be also five times as great; or,

If the velocity is constant, *the momentum is proportional to the mass.*

Again, suppose two bodies of the same mass, but one moving with twice the velocity of the other, its momentum will be also twice as great; or, in general,

If the mass is constant, *the momentum is proportional to the velocity.*

The UNIT OF MOMENTUM is the momentum of a body of unit mass, moving with the unit velocity of one foot per second. A body whose mass is M and whose velocity is

v has a momentum equal to the product of the mass into the velocity, or

$$\text{Momentum} = Mv.$$

EXAMPLES.

XI. *Mass—Density—Volume.* Article 56.

1. The masses of two bodies are as 2 to 7, and their densities as 6 to 5: What is the ratio of their volumes?

2. The masses of two bodies are as 5 to 6, their volumes as 2 to 3: What is the ratio of their densities?

3. Two bodies of the same mass have densities as 8 to 9: What is their ratio in volume?

4. What is the ratio in volume of a piece of silver weighing 20 lbs. and having a density of 10.5 (referred to water as unity), and a piece of iron weighing 5 lbs. and having a density of 7?

5. If a cubic foot of water weighs 62.5 lbs. (density unity), what is the weight of a cubic inch of mercury, density 13.6?

6. What is the ratio in weight (that is, in mass) of two blocks of stone, one having a volume of 50 cubic feet and a density of 3, the other a volume of 45 cubic feet and a density of 2.75?

7. If a liter (1000 cubic centimeters) of water weighs a kilogram (density unity, temperature 4° C. = 39.2° F.), and a cubic centimeter of another liquid weighs 1.1 grams, at the same temperature, what is the density of the latter liquid?

8. If a cubic foot of fresh water weighs 62.5 lbs., and of salt water 64 lbs., what is the density of the salt water?

Kinds of Forces—Force of Gravity.

58. Definition of Force. *A force is that which moves or tends to move a body, or which changes or tends to change its motion, either in direction or quantity.*

In view of the fact, before explained (11), that all bodies of which we have any knowledge are in motion, the completeness of the definition would not be im-

paired by the omission of the first clause. It is, however, convenient to consider the earth and all bodies which do not change their position with reference to it as at rest, and the definition conforms to that idea. Further, the word "tend" is added because the action of one force may be neutralized by that of one or more opposing forces, so that the motion which it *tends* to produce is not observed. For example, a book resting on a table tends to fall to the ground under the action of the force of gravity, but an equal opposite force, the resistance or reaction of the table, keeps it at rest.

59. Continued and Impulsive Forces. A force is said to be *continued* when its action continues an appreciable length of time. It is *uniformly continued*, or *constant*, when its intensity is always the same; this is true of the force of gravity at a given point on the earth's surface. A continued force is *variable* when its intensity is different at different times, as the force of a watch-spring, whose intensity diminishes as the spring unwinds.

An *impulsive* force is one which acts through so short a time that the law of its action cannot be determined, and we are limited to considering its effects after its action has ceased. This is true of the blow from a bat on a ball. Such a force, however, is not strictly instantaneous, but one whose intensity is very great and varies during the brief time of its action. There is consequently no essential difference between the two classes of forces.

60. Effects of Force upon a Free Body. A continued force tends to produce accelerated motion in the body acted upon. If the force is uniform as well, it tends to

give the body uniformly accelerated motion. This is practically the motion of a body falling toward the earth under the influence of the nearly constant force of gravity (but see Art. 64). Therefore, if the acceleration produced by a constant force is f (for gravity g), the relations between the velocity acquired (v) and space passed over (s) in a given time (t) are expressed by the familiar equations (from Art. 27):

$$v = ft, \qquad \text{For gravity, } v = gt,$$
$$s = \tfrac{1}{2}ft^2, \qquad s = \tfrac{1}{2}gt^2,$$
$$v^2 = 2fs. \qquad v^2 = 2gs.$$

The motion of a body acted upon by an impulsive force tends, after this force has ceased acting, to be uniform, as a ball struck along the ground by a bat. This is true, indeed, of any body in motion, and, as explained in Art. 67, is a consequence of the first law of motion.

The presence of other opposing forces may modify the effect of the force considered. For example, a stone thrown vertically upward, and which consequently tends to move uniformly in that direction, has in fact retarded motion because of the continued and simultaneous action of the force of gravity downward. So, too, a ball rolled along the ground, as a matter of experience, soon comes to a state of rest because of the opposing force of friction.

61. Equilibrium. A body is said to be in equilibrium, with respect to two or more forces, when they neutralize each other so that its condition of rest or motion is not affected by them. The book mentioned in Art. 58 is an example of equilibrium. But equilibrium does not

necessarily imply the rest of the body in question. For example, a ball rolling on a perfectly smooth horizontal surface is in equilibrium with respect to the two equal and opposite forces—the action of gravity and the reaction of the surface. The same would hold true however many forces were involved if they, taken together, did not affect the motion of the body. Equilibrium strictly implies simply absence of acceleration.

62. Examples of Forces. The first and simplest conception of force we derive from muscular exertion, as we note its effects in different ways. With it we join all other agencies which produce similar effects in changing the motion of bodies, as gravity, cohesion, electrical attraction and repulsion, and so on.

63. Force of Gravity. The force of gravity is manifested in the attraction which the earth exerts on a mass of matter near its surface, and which causes it to fall, or tend to fall, toward it. This is a special case of the universal law of gravitation, established by Newton, and according to which

Every particle of matter attracts and is attracted by every other particle with a force which varies directly as the product of the masses and inversely as the square of the distance. $F \propto \frac{MM'}{d^2}$.

A stone, therefore, as truly attracts the earth as it is attracted by it; so, also, the earth attracts and is attracted by the moon, the sun, and the other bodies of the solar system. In terrestrial mechanics, however, we have to do simply with the attraction of the earth upon bodies on or near it, and as its mass is indefinitely great in comparison, the reciprocal attraction is left out of

account. Therefore, since the mass of the earth is constant, the force of its attraction on any body varies directly as the mass of that body and inversely as the square of its distance from the earth's centre; that is, $F \propto \frac{M}{d^2}$. From this, it follows that for a given body the force of attraction varies inversely as the square of the distance; and for different bodies at the same distance the force is directly as their masses. These two points are expanded in articles 64 and 65.

The attraction between different bodies, although a force of small intensity where their masses are small, may be demonstrated in other ways than by the fall of a body to the earth. It is illustrated by the deviation from the usual perpendicular position, which is observed when a ball hung by a string is suspended near an isolated mountain. Experimenting in this way, Dr. Maskelyne found the angle of deviation for two plumb-lines, placed on opposite sides, north and south, of Mt. Schehallien in Scotland, and at a distance of 4000 feet from each other, to be 12 seconds. From this result the mean density of the earth was calculated to be about 5 times that of water.

This attraction has also been shown by Cavendish in a more delicate manner, by means of the torsion balance. Two small balls of lead were attached to the ends of a slender wooden rod which was supported at the centre by a fine wire of considerable length. Two larger balls were then approached to the small ones and on opposite sides, so that their effect was felt in the same direction. The result was that the small balls were attracted by the larger ones, and the rod supporting them deflected from its original position of rest. The angle of deflection showed the amount of the torsion (or twist) of the wire, and this measured the intensity of the attracting forces. By comparing the attractions of these balls with that of the earth, Cavendish calculated the mean density of the earth to be 5.45.

64. (1) *The force of attraction of the earth on a given body varies inversely with the square of the distance.*

$F \propto \frac{1}{d^2}$. This law means that if the distance from the attracting body be increased two, three, or ten times, the force of attraction it exerts on another body is $\frac{1}{4}$, $\frac{1}{9}$, or $\frac{1}{100}$ respectively; if the distance is diminished to $\frac{1}{2}$, $\frac{1}{3}$, and so on, the force of attraction becomes 4 times, 9 times, etc., greater.

Two consequences, of importance here, follow from this principle:

(*a*) The attraction of the earth is sensibly constant at a given locality and for the different heights above the surface involved in ordinary observations.

This attraction may be proved to be exerted as if the whole mass were concentrated at a point at or near the centre, called the *centre of gravity*. If now we call the radius of the earth in round numbers 4000 miles, the distances from this centre for two bodies—the one at the sea-level and the other a mile above—will be 4000 and 4001 miles respectively, and the ratio of the forces of attraction will be, as above, $\overline{4001}^2 : \overline{4000}^2$; but this difference is so small that it may often be left out of account. In accurate physical investigations, however, this difference can by no means be neglected. Indeed, it should be most carefully noted that while gravity is here said to be *sensibly* constant, it really varies uninterruptedly as the distance from the centre increases, and is not absolutely the same for two points, one of which is a foot above the other.

If we imagine a body to pass from the surface toward the centre of the earth, it may be demonstrated that the attraction will diminish, and if the earth were homogeneous, in the same ratio as the distance from the centre diminishes; at the centre the attraction is zero.

(*b*) The force of attraction is least at the equator and increases toward the poles.

The earth having the shape of an oblate spheroid, the polar diameter is about 26 miles shorter than the equatorial, and hence, as it may be demonstrated, the force of attraction is greater for a body at the poles by $\frac{1}{570}$.

To this cause for the variation of the intensity of the force of gravity is to be added a second, the rapid rotation of the earth on its axis. The effect of this cause to diminish the gravitation is greatest at the equator, and grows less as we go from it, and becomes zero at the poles (80). On this account, then, the attraction is greater by $\frac{1}{289}$ at the poles than at the equator. As the result of the two causes taken together, the force of gravity at the poles is about $\frac{1}{192}$ greater than at the equator. Thus the *weight* of a given mass of matter, by which the pull of the earth may be measured, increases as we go from the equator northward or southward toward the poles. This difference could be noted, for example, by observations with a sufficiently delicate spring-balance, and would amount to about 1 lb. in 200 lbs. This difference is left out of account in ordinary commercial transactions, but cannot be neglected in physical problems where accuracy is required.

The force of gravity also varies somewhat for different points on the earth's surface in consequence of variations in the density of the material of the earth. This subject is expanded in a later article (250), where the values of g for different points are also given.

65. Again: (2) *The attraction of the earth on bodies at the same distance is proportional to their masses.* $F \propto M$. Hence the acceleration given to a falling body by the earth's attraction is independent of its mass.

For example, two pieces of lead of widely differing weights, or a piece of lead and a feather, fall the same distance toward the earth in the same time and gain the same velocity (that is, 32 feet per second for each second). But the experiment succeeds only when all disturbing causes—*e.g.*, the resistance of the air—are removed, so that an exhausted receiver must be employed. As a matter of experience, in comparing the fall of a small and heavy body and of a larger and lighter one, the former will fall the faster under ordinary conditions, but this is only because it feels less the resistance of the air.

The fact here mentioned is based upon the above law, and is what a simple consideration would lead us to expect. Suppose several small shot as those at *a* (Fig. 28); it is obvious that under the earth's attraction they would fall together, keeping their relative position with reference to each other, and reaching the ground at the same time and with the same velocity. If now we suppose them rigidly connected, but their positions unchanged, they would form a mass 5 times greater than a single one (as *b*), and attracted by a force also 5 times greater; they would still fall together, and the acceleration of this mass and the single one would be the same. This course of reasoning might be extended to any two bodies, however unlike in mass. The force of attraction increases always in the same ratio that the number of particles, or the mass, of the body attracted increases, and hence the effect of the force of attraction must remain constant.

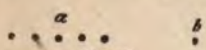


Fig. 28.

EXAMPLES.

XII. *Force of Gravity.* Articles 63, 64, 65.

1. At what distance from the centre of the earth would a mass of matter weighing 32 lbs. on the earth's surface exert a pull equivalent to 1 lb. on a spring-balance ?

2. If the mass of the sun is 350,000 times that of the earth, and its diameter 112 times, what is the acceleration of gravity at its surface ? (see also p. 67)

3. If the moon's mass is $\frac{1}{80}$ of that of the earth, and its diameter 2160 miles, that of the earth being about 7900 miles, what is the acceleration of gravity on the moon's surface ?

4. If the acceleration of gravity on the surface of Jupiter is 2.62 times g, and its diameter 11 times that of the earth, what is their ratio in mass ?

5. If the value of g at the equator, at the sea-level, is 32.096, what is its value at the summit of a mountain in latitude 0°, at an altitude of 15,840 feet ? The equatorial radius is 3963.3 miles.

Newton's Laws of Motion.

66. Laws of Motion. The three laws of motion, as stated by Newton, are:

(1) Every body continues in a state of rest, or of uniform motion in a straight line, except in so far as it may be compelled by impressed forces to change that state.

(2) Change of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.

(3) To every action there is always an equal and contrary reaction; or, the mutual actions of two bodies are always equal and opposite in direction.

The truth of these laws is established by observation and experiment; it is found that all legitimate conclu-

sions deduced from them are in harmony with the observed facts of nature.

67. The FIRST LAW asserts what is sometimes called the *inertia* of matter; in other words, that matter alone is powerless to change its state either of rest or motion. These points may be considered separately.

(*a*) The tendency of a body at rest to remain in this condition is universally recognized; it is always manifested by the apparent resistance which such a body offers to a force tending to set it in motion. This apparent resistance of heavy bodies has nothing to do with that caused by other opposing forces; *e.g.*, friction. It increases, as stated in 53, with the mass of a body, and, as explained under the second law of motion (68), is due solely to the fact that in such cases a force must continue to act through a certain time in order to impart sensible motion. For example, suppose a heavy fly-wheel of an engine free to turn on its axis without friction, or a heavy cannon-ball suspended by a string of great length, or a massive iron door well poised on its hinges; in all such cases the hand in trying to move the bodies seems to encounter resistance, which expresses the inertia of the body, or its tendency to remain at rest. The only real difference, however, between the bodies named and a light body moved with a touch arises from the greater mass involved in the former case; both alike have inertia.

This inertia, or resistance to motion, is a property of all forms of matter; it is manifested by the water when a boat moves rapidly through it, and by the air when one is driving or running rapidly.

(*b*) The application of the second part of the law is equally familiar, although it is impossible to give an

experimental proof of its truth. It requires that a body once in motion shall—unless acted upon by some force tending to change its motion—continue to move forever uniformly and in a straight line. This tendency toward continuance in a state of motion is seen in the apparent resistance that bodies in motion make to a force tending to stop them. We observe, also, that if a carriage in rapid motion is suddenly stopped, the persons occupying it are thrown violently forward; so, too, if a person steps off from a train in rapid motion, his body tends to keep its forward motion, and when the motion of the feet is arrested by touching the ground, the rest of the body is thrown forward and a fall is the result.

But a body in motion not only tends to move on uniformly, but also in *a straight line*. Hence if we observe a body moving in a circular path, as a stone attached to a string and whirled about a centre, we are justified in concluding that a force is continually acting to deflect it from a straight line. If the string to which the stone is attached breaks, the stone flies off at a tangent to its former course. So, too, of the earth: it moves about the sun in a path nearly circular, its tendency to move in a straight line being overcome by the continued attraction of the sun.

The *tendency* to uniform motion in a right line is all that can be observed; for, as stated above (60), uniform motion in a body not acted upon by any force is altogether contrary to experience—any body, once set in motion, sooner or later comes to rest. But in every such case it is possible to trace the more or less rapid loss of motion to outside causes; the most universally present are friction, or the resistance to motion due to the rough-

nesses of the surfaces in contact, and the resistance of the air, which last is an important element in the case of rapid motion, as that of a bullet.

The truth of the law is argued from the observed fact that, in proportion as these opposing forces are removed, the motion continues longer and longer. A ball, which with a given impulse will roll a certain distance on a horizontal surface of turf, rolls farther on gravel, farther still on a marble floor, and still farther on a sheet of ice. So, too, the time which a pendulum, once set in motion, will continue to vibrate without additional impulse becomes longer and longer as we remove the friction on its axis of support, and the resistance of the air by placing it in an exhausted receiver. The case of the earth is the most perfect illustration of this law, for it moves on in its orbit with a mean velocity that the most accurate observations can hardly prove to vary, although it is receiving no forward impulse; the sun's attraction, as explained above, only serves to keep it in its nearly circular orbit.

68. The SECOND LAW of motion asserts (a) that the change of motion is proportional to the impressed force.

By motion is meant here *momentum*, or the product of the mass and velocity, as defined in Art. 57. The law consequently asserts that the change of momentum in a given time is proportional to the force which acts, and hence, as explained in 71, this change of momentum is a measure of the force.

Let F represent the force, M the mass of the body moved, and f the velocity given to it in one second (the acceleration). Then Mf is the momentum generated in one second, and by this law F is proportional to Mf; that is, F is equal to Mf multiplied by some con-

stant number; if suitable units are taken, this constant becomes unity, and then

$$F = Mf. \qquad (1)$$

If the force acts for t seconds with uniform intensity, then its effect will be proportional to the time ($= Ft$), and the velocity given to the body at the end of this time will be ft or v. Hence, with the same provision as above, we have

$$Ft = Mv. \qquad (2)$$

In the case of gravity the momentum generated in a given time—that is, in one second—is Mg. But since the weight of the body, expressed in standard pounds, is proportional to the mass (M), and also to the acceleration of gravity (g), the weight is also proportional to the product Mg; if a suitable unit of mass is taken, we may write

$$W = Mg, \qquad (3)$$

or

$$M = \frac{W}{g}. \qquad (4)$$

The expression in equation (4) is the value of the mass in terms of the weight in pounds which is ordinarily employed in Mechanics.

From equation (2) the following principles are deduced:

1. *The velocities given in the same time to different bodies of the same mass are proportional to the acting forces.* That is, if forces whose intensities are as 1, 2, 3, act on three bodies of equal mass, the velocities generated in the same time will be in the ratio of 1 : 2 : 3. For example, the intensity of the sun's attraction at its surface is about 28 times, and of Jupiter 2.6 times, that

of the earth; therefore the velocity acquired at the end of one second by a falling body, on the sun, on Jupiter, and on the earth, will be respectively 28×32 feet per second, 2.6×32, and 32.

2. *If equal forces act upon bodies of different mass for the same time, the velocities will be inversely proportional to the masses.* A force which would give a body of mass 1 a velocity of 12 feet per second in a certain time, would give a body of mass 3 a velocity of only 4 feet per second in the same time. This principle shows the reason why a "heavy" body—that is, one of great mass—seems to offer more resistance to motion than a lighter one. For suppose the ratio in mass to be 10 : 1; then the same force acting upon the heavier body will give it in the same time the same momentum, but only $\frac{1}{10}$ the velocity. In other words, it must act through ten times the length of time in order to generate the same velocity.

Again—3. *To give bodies of different mass the same velocity in the same time, the forces must in each case be proportional to the masses.* As explained in 65, this is true of the force of the earth's attraction, which gives to all falling bodies at its surface the same acceleration.

Finally—4. *If equal forces act upon bodies of equal mass, the velocities generated will be proportional to the times of action.*

(*b*) The second law also states that the direction of motion is that of the impressed force. When a force acts upon a material particle at rest—that is, a portion of matter so small that its dimensions may be left out of account—the truth of this law is evident. If a body is acted upon, it will in general take the direction of the impressed force only when the line of its action passes

through a point in it called the centre of inertia. When this is not the case, as when a ball is struck upon its side by a bat, there is a tendency to rotate, and the body moves in a direction which varies more or less from that of the force.

In general, if any number of forces act upon a body, this law will hold good for each of them as though the others did not exist; or the effect upon the body, as regards quantity and direction of motion, at the end of any time, is the same as if the forces had acted each through the same time in succession.

So far as it relates to the direction and rate of motion this is properly the basis of the Parallelogram of Velocities, explained in Art. 33. For example, in the crossing of the stream by the uniformly moving boat (36), each motion goes on independently, not modified by the existence of the other, although the position of the boat at any moment depends on the action of both. Again, every motion goes on aboard a steamboat indifferently whether the boat is or is not in motion: a ball thrown up by a person standing on deck falls again to his hand, and if dropped from the top of the mast falls at its foot a second or two after, although the motion of the boat may have been rapidly forward all the time. So, too, a rifle-ball fired horizontally from a window, and another dropped vertically down at the same instant, reach the ground in the same time (allowance being made for the resistance of the air). In each case the forward and the downward motions go on independently of each other.

69. The THIRD LAW is sometimes briefly stated in this form, that: *The action and reaction are equal and contrary.*

The general term *stress* is employed to express the mutual action between two bodies, for the action of a force always implies *two* bodies, and this law affirms that the action of the one is exactly equal to the reaction of the other.

This law is true whether the force is exerted as pressure, as a pull or tension, or as a blow.

(*a*) *Pressure.* For instance, when the hand presses against the wall, a contrary and equal reaction is exerted by the wall. A weight, exerting pressure on a support, encounters an equal return pressure upward.

(*b*) *Tension.* For example, a ten-pound weight, hanging by a string, exerts a pull of 10 lbs., and the reaction of the hook to which the string is attached is also 10 lbs. Also, if a horse drags a heavy load by a rope, the load pulls back an amount equal to that which the horse exerts.

(*c*) *Blow.* When a hammer strikes a nail, the reaction is equal and contrary. Again, when a cannon is fired off, the recoil of the gun and carriage is due to the reaction equal to the action in driving forward the shot.

The third law of motion, taken in its broadest sense, is true only when the apparent loss of mechanical energy on impact is allowed for, as explained later (108 *et seq.*).

70. Collision of Inelastic Bodies. It follows from the preceding article that: *When several bodies come into direct collision, the momentum of the whole system before and after impact is the same.*

Suppose two *inelastic* bodies whose masses are M and m, and whose velocities are V and v; the momentum of the first is MV, and of the second is mv. (*a*) If they are moving in the same direction, the momentum of the two is

$$MV + mv.$$

If now the faster-moving body overtakes and impinges upon the other, the two after impact will move along together with a velocity v' less than V and greater than v; the momentum of the two together will be

$$(M + m)\, v',$$

and by this law

$$MV + mv = (M + m)\, v'. \qquad (1)$$

(*b*) If the two bodies were moving in opposite directions, then the momentum of the two moving separately is

$$MV - mv,$$

and after impact of the two together,

$$(M + m)\, v';$$

and, as before,

$$MV - mv = (M + m)\, v'. \qquad (2)$$

As an example of this, suppose that a rifle-ball weighing one ounce and moving with an unknown velocity v strikes and penetrates a body whose weight is 10 lbs., and that after impact the velocity of the mass of wood with the imbedded ball is $v' = 8$ feet per second; then

$$\tfrac{1}{16} \times v = (10 + \tfrac{1}{16}) \times 8.$$

From which it may be calculated that $v = 1288$. This is the principle involved in the *ballistic pendulum;* the velocity after the impact is determined, however, not directly but by calculation from the height to which the mass is raised ($v^2 = 2gh$).

If the impinging bodies are perfectly or imperfectly elastic, the conditions are changed, and a factor expressing the degree of elasticity (coefficient of elasticity) must be introduced. The discussion of these cases lies outside of the scope of the present work.

EXAMPLES.

XIII. *Collision of Inelastic Bodies.* Article 70.

[The bodies are supposed to be perfectly inelastic, and their motion is uniform; the impact is direct, not oblique.]

1. A ball weighing 10 lbs. and having a velocity of 16 feet per second overtakes a second ball weighing 5 lbs. and whose velocity is 8 feet per second: What is the final velocity?

2. If the first ball in the preceding example meets the second, what is the final velocity?

3. A body weighing 40 lbs. strikes another at rest weighing 360 lbs., and the two move on with a velocity of 2 feet per second: What was the original velocity of the first ball?

4. Three bodies, each weighing 4 lbs., are situated in a straight line; a fourth, weighing 8 lbs. and moving at a rate of 12 feet per second, strikes them in succession: What velocity results after each impact?

5. Two bodies moving in the same direction at the rates of 8 and 10 feet per second come into collision, and after impact have a velocity of 8.4 feet per second: What is the ratio of the masses of the two bodies?

6. If the bodies in example 5 move in opposite direction, and the final velocity is .4 feet per second, what is the ratio of their masses?

7. A body moving 10 feet per second meets another moving 2 feet per second, and thus loses one half of its momentum: What is the ratio of the masses of the two bodies?

8. A body weighing 6 lbs. strikes another weighing 5 lbs. and moving in the same direction at a rate of 7 feet per second: If the velocity of the second body is doubled by the impact, what was the previous velocity of the first body?

9. An ounce rifle-bullet is fired (as in 70) into a suspended block weighing 36 lbs.; the blow causes the wood to rise $1\frac{1}{2}$ inches: Required the velocity of the bullet at the moment of impact.

Measurement of Force.

71. Absolute Method of Measuring Force. A force may be measured: *By the velocity which it gives the unit*

of mass in the unit of time. The UNIT FORCE is then a force which will give a pound of matter a velocity of one foot per second in a second. This unit is sometimes called a *poundal.* It may also be stated in this equivalent form, already implied in Art. 68: A unit force is one which will generate (or destroy) a unit of momentum in one second.

When the units of the metric system are employed, a unit force is defined as one which will give one gram of matter a velocity of one centimeter per second in one second; this unit force is called a Dyne. 13,825.38 dynes make one poundal. This system of measuring force is called the centimeter-gram-second system, or the C.G.S. system.

The force of gravity on a pound of matter, which gives a velocity of about 32 feet per second in a second (g), is then a force of 32 poundals, and hence this number 32 (or g) is the measure of the earth's attraction on this absolute system. As it is 32 times the force required to give one pound a velocity of one foot per second, it is evident that the unit force, the poundal, is equivalent to the action of gravity on about half an ounce $\left(\frac{1 \text{ lb.}}{32}\right)$.

This method of measuring force is called *absolute measure,* in the sense that it is universally applicable and independent, as the following method is not, of the variations in the force of gravity. As implied above, the UNIT OF MASS in this system is the standard pound.

72. Gravitation Method of Measuring Force. Forces are also measured: *By comparing them directly with gravity; that is, by the weights they could support.* The UNIT FORCE is then a force equal to that required to support the standard pound against the force of gravity; or, briefly, it is equal to the weight of one

pound. It is then g times, or about 32 times, the unit of force mentioned in the preceding article. This is called *gravitation measure.*

Since now the force of gravity manifests itself everywhere and at all times on the earth, and since we are so familiar with its intensity as measured by the weights of one, two, ten pounds, and the amount of muscular exertion required to overcome it, this is a most simple and natural way of measuring all forces. After this method we say that the tension of the rope pulling a canal-boat is 100 lbs. when the force exerted is equal to that required to support a weight of 100 lbs. It is common to speak of a pull—as on an oar—of 50 lbs., of the force of the wind or that of the waves as being so many pounds, etc. In cases like the last a dynamometer is employed, and the pressure on a spring noted, and this readily compared with the same effect produced by a known weight under the action of gravity.

Notwithstanding the fact that this method is so commonly and conveniently employed, it is less scientific than the absolute method, and is open to one serious objection, that it does not necessarily take into account the variations in the force of gravity. As explained in Art. 64, the force of gravity varies about $\frac{1}{200}$ between the equator and the poles, and, when the same mass is used as the unit of weight, a pull of a pound means a stronger pull in high latitudes than toward the equator. Hence the gravitation method is accurate only when the difference in the value of g at the spot in question and at the sea-level is known and taken into account.

When forces are considered as producing accelerated motion, the absolute measure is generally employed; but when they act, as usually in Statics, either as tensions or

pressures, the gravitation measure is the one usually accepted.

The UNIT OF MASS employed (in the gravitation measure of force) is the quantity of matter in a body which weighs g pounds, where g is the acceleration of gravity for the place in question. This is then a varying unit, having a definite value for each place under consideration. The reason for selecting this unit is that the numerical expression of the mass of a given body will be the same everywhere; that is, a body of mass 10 will still be this wherever it is. For since, as has been shown (68), we may take $M = \frac{W}{g}$, therefore this ratio of $\frac{W}{g}$, and hence the numerical value of M, will be always the same. For example, on the sun, where the force of attraction is about 28 times that on the earth, we should have

$$M = \frac{28. W}{28.g},$$

or the same value as before.

Problems in Dynamics.

73. In Art. 68 it was shown that the intensity of any acting force (F) is equal to the product of the mass moved (M) into the velocity generated in one second; that is,

$$F = Mf. \qquad (1)$$

If in this equation we substitute the value of $M\left(=\frac{W}{g}\right)$,

we obtain
$$F = \frac{W}{g}.f, \qquad (2)$$

or
$$f = \frac{F}{W}\cdot g. \qquad (3)$$

This relation makes it possible to obtain the acceleration (f) produced by any known force acting upon a body whose weight is also known. When f is known, then the equations of articles 27, 41, 42 give the means of calculating all the particulars in regard to the motion of the body; that is, the space passed through in a given time, the velocity acquired, etc.

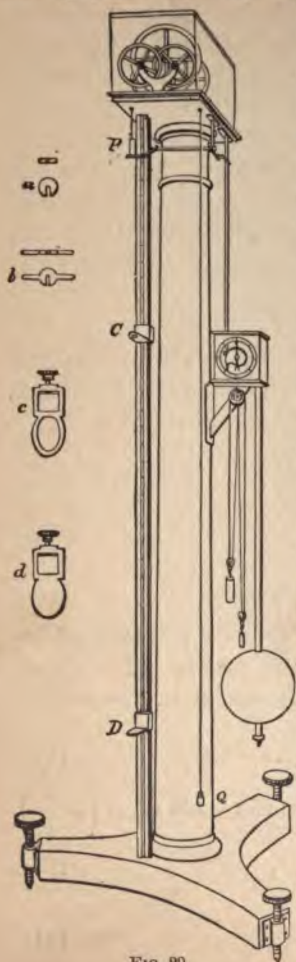


FIG. 29.

74. Attwood's Machine. Equation (3) in the preceding article, in connection with Attwood's machine, makes it possible to verify the laws of motion given in articles 67, 68. The essential parts of the machine are shown in Fig. 29. There is a pulley over which a thread passes, holding any weights P and Q. The axis of the pulley rests on two smaller wheels, called *friction-wheels,* which serve to diminish the friction so that its effect can be disregarded. The whole is supported by a firm, massive stand. In the path of the weights may be clamped a stage (as d) at any point, as determined by the vertical scale D. A ring (c)

in the same position serves, if desired, to remove a portion of the weight without disturbing the motion of the rest. For this last purpose the extra weight is in the form of a straight bar (b), which is caught by the ring as the weight moves through it. A pendulum to beat seconds is added to the whole.

The following are some of the experiments which may be tried:

(1) Suppose the weight $P = 8\frac{1}{4}$ oz. and $Q = 7\frac{3}{4}$ oz., supported on either thread; then the weight moved (W) is $= P + Q = 16$ oz. ($= 1$ lb.), and the moving force (F) is the difference, or $P - Q = \frac{1}{2}$ oz. [Strictly the weight moved also includes the weight of the pulley, which in an accurate experiment would have to be taken into account.] Therefore, by equation (3), if $g = 32$,

$$f = \frac{F}{W} \cdot g = \frac{P - Q}{P + Q} \cdot g = 1 \text{ foot-per-second per second.}$$

Now, by Art. 27, the space passed through in the first two seconds ($t = 2$) by a body moving with uniformly accelerated motion is $2f$. Clamp the stage at a distance of 2 feet below the initial point of P, and the weight descending will strike the stage exactly two seconds after starting. This is also a verification by experiment of the remark, made in Art. 71, that the unit force, the poundal—that is, the force which would give the unit of mass 1 lb. (here $P + Q$) in a second a velocity of one foot per second—was equal to the weight of $\frac{1}{2}$ oz. ($= P - Q$).

If the stage be clamped at $4\frac{1}{2}$ feet, then the weight will strike it at the end of three seconds, as the formula $s = \frac{1}{2}ft^2$ requires. Showing, too, that in three seconds the space passed through is $\frac{9}{4}$ times that in two seconds;

or, in other words (27), the *space is proportional to the square of the time* ($2^2 : 3^2 = 4 : 9$).

(2) Again, let $P = 8.5$ oz. and $Q = 7.5$ oz.; then $P + Q = 16$ oz., or the weight moved is the same as in (1), but the moving force $P - Q = 1$ oz., or twice that in (1); then, as before,

$$f = \frac{P - Q}{P + Q} \cdot g = 2 \text{ feet-per-second per second.}$$

Hence, in accordance with the law stated in Art. 68, the *mass remaining constant, the velocity generated in a given time is proportional to the force acting;* that is,

$$\frac{F}{F'} = \frac{f}{f'} = \frac{1}{2}.$$

If, as before, the stage is clamped 4 feet below the starting-point, the weight will strike it at the end of two seconds, as the formula requires ($s = \frac{1}{2}ft^2$).

(3) Let $P = 4\frac{1}{4}$ oz. and $Q = 3\frac{3}{4}$ oz.; then $P + Q = 8$ oz.; that is, the weight (or mass) moved is one half that in (1), while $P - Q = \frac{1}{2}$ oz.; that is, the moving force is the same. Equation (3) gives

$$f = \frac{P - Q}{P + Q} g = 2 \text{ feet-per-second per second.}$$

That is, in accordance with the law stated in Art. 68, *the acting force being constant, the velocity generated in a given time is inversely as the masses acted upon.*

$$\frac{f'}{f} = \frac{m}{m'} = \frac{16}{8} = \frac{2}{1}.$$

This value of f may be verified as before.

(4) Let $P = 8.5$ oz. and $Q = 7.5$ oz., and let the excess of P over Q (1 oz.) be in the form of a rod pro-

jecting so as to be removed by the ring. Then, for this case, $f = 2$; hence if the ring be placed one foot below the starting-point, the weight will reach it at the end of one second. Here the extra weight is left behind, and now, the two weights being equal, the bodies must, according to the first law of motion, move on with uniform velocity. At the end of another second it will strike the stage if placed 2 feet below—that is, 3 feet from the starting-point—and at the end of two seconds at 5, and so on.

75. The following is a similar application of the above principle. Let W (Fig. 30) be a weight resting on a perfectly smooth horizontal plane; P is another weight attached to W by a string passing over the pulley a; then, neglecting the weight of the pulley, when P falls it moves W also; hence the total weight moved is $P + W$, and the moving force is P. Hence equation (3) becomes

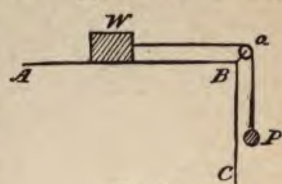


FIG. 30.

$$f = \frac{P}{P + W} \cdot g.$$

The *tension* (T) of the string is given from the relation

$$\frac{T}{W} = \frac{f}{g};$$

substituting in this the above value of f, we obtain

$$T = \frac{PW}{P + W}.$$

76. The relation (3) in Art. 73 is applicable to a retarding force such as friction. For example, let W be a weight moving on a rough horizontal plane; the force of a friction (F) is a force acting parallel to the surface in a direction opposite to the motion. The retardation due to friction is given by the equation

$$f = \frac{F}{W} \cdot g.$$

If F is known, and also W, then f can be calculated; this is the *retardation* due to friction; in other words, the body loses each second in velocity f feet per second. The distance which the body will go through before coming to rest, and its distance at any moment from the starting-point, will be given by the formulas

$$V = u - ft,$$
$$s = ut - \tfrac{1}{2}ft^2.$$

Other illustrations are given in the examples below.

XIV. *General Dynamical Problems.* Articles 68, 73–76.

1. If in Attwood's machine $P = 4\frac{1}{2}$ oz. and $Q = 3\frac{1}{2}$ oz.: (*a*) What is the acceleration? (*b*) What space will be passed through in 2 seconds?

2. (*a*) At what height above the earth's surface would a body fall 4 feet in the first second from rest? (*b*) If its weight was 40 lbs., what pull would it exert on a spring-balance at this point?

3. A 12-lb. weight hanging over the edge of a smooth table drags a 60-lb. weight with it: What is the acceleration and the tension of the string?

4. If the table in example 3 is rough, and the resistance of friction consequently equivalent to one tenth of the weight of the sliding body, what is the acceleration?

5. (*a*) A bucket weighing 100 lbs. is raised up from a well at

a uniform rate of 12 feet per second: What is the tension of the rope? (*b*) If the acceleration is 4 feet-per-second per second, what is the tension?

6. For what time must a force of 4 oz. (gravitation measure) act on a body weighing 8 lbs. to give it a velocity of 20 feet per second?

7. Two weights of 16 and 14 oz. hang over a pulley: What space will they move through from rest in 3 seconds?

8. Two weights, each 8 oz., hang over a pulley: What additional weight must be added to one of them to give an acceleration of 2 feet-per-second per second?

9. A weight of 8 lbs. rests on a smooth horizontal table 12 feet wide: What weight hanging vertically will draw it across in 3 seconds?

10. A weight of 24 lbs. rests on a platform: (*a*) What is its pressure on the platform if the latter is ascending with an acceleration of $\frac{1}{8}g$? (*b*) If descending with the same acceleration?

11. A body weighing 160 lbs. is moved by a constant force, which generates a velocity of 8 feet per second: What weight could the force support?

12. A force of 8 lbs. (gravitation measure) acts constantly on a body weighing 24 lbs. and resting on a smooth horizontal surface: (*a*) What is the acceleration, and (*b*) how far will the body move in 4 seconds?

13. The velocity of a body weighing 24 oz. is increased from 20 to 40 feet per second while the body passes over 30 feet: What is the moving force?

14. Of two weights hanging over a pulley, one is 1 lb. and it ascends with an acceleration of 10 feet-per-second per second: What is the other weight?

15. How long must a constant force of 10 lbs. act on a mass of 100 lbs. to give it a velocity of 30 miles an hour?

16. What constant force will cause a body weighing 400 lbs. to pass over 1200 feet in 10 seconds from rest on a smooth horizontal surface?

17. A constant force of 15 lbs. gives a body an acceleration of 5 feet per second in one second: What is the weight of the body?

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18. A body weighing 24 lbs. is projected on a rough horizontal

surface, where the resistance of friction is 6 lbs., with an initial velocity of 64 feet per second: (*a*) How far and (*b*) how long will it slide before coming to rest?

19 A body weighing 40 lbs. is projected as in the preceding example. What is the resistance of friction if the body slides 16 seconds before stopping?

20 A body weighing 60 lbs. is projected up a rough plane inclined at an angle of 30°, where the resistance of friction is 6 lbs., and with an initial velocity of 160 feet per second: (*a*) How far on the plane will it ascend? (*b*) How long will it take?

21. The length of an inclined plane is 1000 feet, and its base 800 feet; the resistance of friction is one eighth of the weight: (*a*) What initial velocity must it have just to reach the top, and (*b*) how long will the ascent take?

CHAPTER III.—CENTRAL FORCES.

77. Uniform Circular Motion. Suppose a body to be moving in a circular path with a constant velocity; according to the first law of motion (66), a body in motion, if not acted upon by any force, tends to move on uniformly in a straight line. In order, therefore, that this body should, as supposed, move in a circle, it must be acted upon by a constant force exerted in the direction of the centre of the circle.

This force toward the centre, or central force, is called the *centripetal force.* The equal and opposite (69) reaction exerted away from the centre is called the *centrifugal force.* The central forces determine the *direction* of motion of the body, but do not affect its rate of motion or velocity, since they act continually at right angles to its path. If a body attached to a string be whirled about a centre, the intensity of these central forces is measured by the tension of the string. If the string be cut, the body will fly off in a tangent to the curve, but with unchanged velocity.

78. To find the intensity of the central force in the case of uniform circular motion: Suppose the body to be moving at a constant velocity v about the circle ADH (Fig. 31), whose radius (AC) is r. In a very short space of time (t) it will have gone from A to D, so that

$$AD = vt. \qquad (1)$$

But in this time it has been drawn away from the tangent toward the centre a distance equal to BD (or AE). Let f be the acceleration of the constant central force; then (60)

$$AE = \tfrac{1}{2}ft^2. \qquad (2)$$

But since AFH is a semicircle, the angle ADH is a right angle, and, by geometry,

$$\overline{AD}^2 = AE \times AH = AE \times 2r. \qquad (3)$$

If AD is taken very small, the chord AD may be regarded as identical with the arc AD. Therefore,

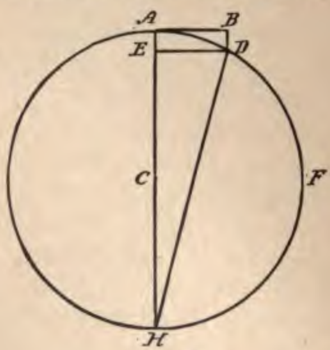

FIG. 31.

introducing into (3) the values given in (1) and (2), we have

$$v^2t^2 = \tfrac{1}{2}ft^2 \times 2r,$$

or

$$f = \frac{v^2}{r}. \qquad (4)$$

This value of f (4) gives the acceleration due to the constant force in terms of the velocity and the radius of the circle. From it we see that the value of f varies

directly with the square of the velocity, and inversely as the radius.

In order to obtain the intensity of the force (F)—that is, in the case supposed above, the tension of the string—the value of f must be multiplied, as explained in 68, by the mass (M) of the body in motion. We have then

$$F = Mf = \frac{Mv^2}{r}.$$

If the value of F in pounds be required, when the weight (W) is given: since $M = \frac{W}{g}$, we have, further,

$$F = \frac{W}{g} \cdot \frac{v^2}{r}.$$

79. The pull away from the centre, called the centrifugal force, is felt whenever a body is made to rotate rapidly about a fixed centre. It is exemplified by the case of a loaded sling: if the cord is elastic, the extent to which it is stretched is a measure of this force. Similarly, a bucket containing water may be swung around by a rope so rapidly that this force becomes greater than that of gravity, and the contents are consequently not lost even when it is inverted.

In the case of a large wheel rotating rapidly, if the centre of gravity and the axis of rotation coincide, the effects upon the parts on opposite sides of the axis neutralize each other and produce no result, except when the force becomes greater than the cohesion between the particles, when fracture takes place—as when a grindstone breaks. If, however, the centre of gravity does not coincide with the axis, a continuous pull on the bearing is produced by the motion which may lead to

very injurious results. For the same reason a crank-arm, which in use is turned rapidly, is generally weighted on the side of the centre opposite to the handle, to neutralize the injurious pull on the axis that would otherwise exist.

If a globe containing a little mercury be set in rapid rotation, the effect is to cause the mercury to recede from the axis and hence to rise and form a ring about the central part farthest from the axis. The governor of Watt, applied to the steam-engine, consists essentially of two heavy balls carried on rods jointed at the top. They are connected with some turning part of the engine, and, on the above principles, an increase in the rate of revolution causes them to separate and rise, and conversely if the rate is diminished. The arrangement is such that in the former case they partially close, and in the other case open, a valve by which the supply of steam is received. They thus serve to regulate the motion and keep it uniform; whence the name the instrument has received.

80. Centrifugal Force due to the Earth's Rotation. Since the earth is rotating on its axis, every body on its surface, tending to move on in a straight line, must be retained there by a force pulling toward the axis. In other words, a certain portion of the earth's attraction for every body on its surface is exerted simply to constrain the body to move in a circle, and is consequently not felt as weight. This is equivalent to saying that the centrifugal force acts on every body directly away from the centre of the circle in which it is moving. The direction of this force is indicated for the points E and B (Fig. 32) by the lines EG and BK.

Since, by the preceding article,

$$f = \frac{v^2}{r},$$

the value of f can be calculated for the equator, for the value of v is given by the fact that any point on it

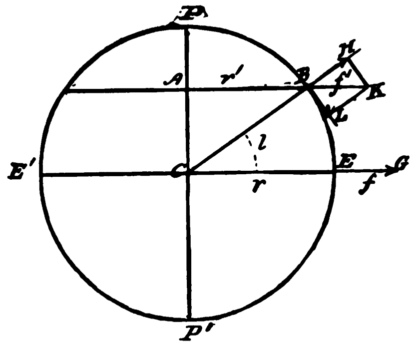

FIG. 32.

describes a distance equal to the equatorial circumference in 24 hours. In this way we obtain

$$f = .1112 \text{ feet-per-second per second.}$$

This value of f is about $\frac{1}{289}$ of the value which g would have at the equator if this influence did not exist; since $17^2 = 289$, it follows that, if the velocity of rotation of the earth were increased 17 times, all bodies upon it at the equator would entirely lose their weight.

81. For any point as B (Fig. 32) where the latitude is l, the value of this force f', and the velocity v' ($= v \cos l$), we have

$$f' = \frac{v'^2}{r'} = \frac{v^2 \cos^2 l}{r \cos l} = \frac{v^2}{r} \cos l,$$

$$\therefore f' = f \cos l,$$

The components of f' (BK, Fig. 31) are BH normal to the surface and BL as a tangent. Of these, since $HBK = l$,

$$BH = f' \cos l = f \cos^2 l;$$

$$BL = f' \sin l = f \sin l \cos l = f \frac{\sin 2l}{2}.$$

The normal component alone influences the weight of the body as it acts directly contrary to gravity, while the tangential component tends to produce motion toward the equator. It was the influence of the tangential component when the earth was in a plastic condition which is believed to have caused the flattening at the poles. This effect may be illustrated by the rapid rotation of a plastic mass of clay on its axis.

The value of f is greatest at the equator and diminishes, with the cosine of the latitude, as we go toward the poles; at the poles it is zero ($\cos l = \cos 90° = 0$). The normal component is greatest at the equator and diminishes with the square of cosine of the latitude. The tangential component is zero at the equator, increases to latitude 45° $\left(BL = \frac{f}{2}\right)$ and diminishes from there to the poles, where it is again zero.

EXAMPLES.

XV. *Centripetal and Centrifugal Forces.* Articles 77–81.

1. A ball weighing 20 lbs. is whirled by means of a string around a centre at a radius of 7 feet, with a linear velocity of 28 feet per second. What is the value of f, and what is the tension of the string (F)?

2. (*a*) If the velocity is doubled in the preceding example, what do the values of f and F become? (*b*) What are they if the radius is doubled?

3 A ball weighing 4 lbs. attached to a centre at a distance of 8 feet makes 300 revolutions in a minute: What is the pull on the centre ?

4. What linear velocity of rotation must a body have if the tension of the string by which it is attached to the centre, at a distance of 8 feet, is equal to its weight ?

5. What is the angular velocity of a body moving in a circle with a radius of 4 feet, when the centrifugal force is one half the weight ?

6. A locomotive weighing 12 tons moves at a rate of 30 miles an hour about a curve whose radius is 1000 feet: What is the horizontal pressure on the rails ?

7. If a stone weighing 5 lbs. is attached to a string 3 feet long and makes two revolutions in a second, what is the pull on the centre ?

8. If a string can just support a weight of 400 lbs., what is the greatest length that can be employed to swing around a 20-lb. weight once in a second ?

9. What is the shortest length of the string only strong enough to support 100 lbs. that can be used to whirl around a 50-lb. weight at a rate of 8 feet per second ?

CHAPTER IV.—FRICTION.

82. Definition of Friction. *Friction is the resistance which is offered to the motion of one body upon another due to the roughness of the surfaces in contact.* Friction always acts parallel to the surfaces, and in a direction contrary to that in which the body is moving or is about to move.

83. Reaction of Smooth Surfaces. The only effect produced by the mutual pressure of two *perfectly smooth* surfaces would be the reaction perpendicular to them at the point of contact. It would hence exert no influence on the motion of one upon the other; in such a case there would be no friction. (There would still be, however, even in this case, resistance to motion due to the mutual adhesion of the surfaces in contact; but this is an independent matter not here considered.)

An ideally smooth surface cannot be obtained, even by continued polishing; hence the resistance of friction can never be entirely eliminated. It is found in general that the smoother the surface is made the less is the friction.

It is obvious, also, that the actual reaction between two rough surfaces which are in motion, or about to move, is in the direction of the resultant of the two components, one the force of friction parallel to the surface, and the other the normal pressure as defined in Art. 89.

84. Since friction is diminished by rendering the surfaces in contact more smooth, it is customary to make use of *lubricators*, which fill up the unevennesses of the surfaces. For example, oils, lard, graphite, soapstone, and other substances are employed. The best lubricator, in a given case, depends upon the materials in contact and the amount of pressure sustained, and is determined by experiment. The friction of an axle may be much diminished by the use of *friction-wheels;* the axle rests upon two wheels, which turn with it, as indicated in Fig. 29 (p. 76) and Fig. 44 (p. 117).

Conversely, when it is desirable to increase friction the surfaces may be made more rough. For example, when the driving-wheels of a locomotive tend to slide on the rails because the latter are wet and slippery, it is customary to feed down sand, by a tube from the sand-box above, on to the rail in front of each wheel, which has the desired effect.

85. Friction is, so far as this, a *disadvantage* from a mechanical point of view, since force is required to overcome it, and this represents so much working power or energy expended without any useful result. For example, in the machinery of a cotton-mill, at every bearing there is friction, and the engine which supplies the energy for the establishment must furnish beyond what is required for the useful work performed enough more to make good this waste. Again, the locomotive on a railroad, after the train is in motion and supposing the track horizontal, exerts its energy solely against the outside resistance, which is chiefly caused by the friction of the axles in their bearings and the wheels on the rails.

On the other hand, however, friction is often mechani-

cally an *advantage*. It alone gives to the driving-wheels of the locomotive their hold on the track; without it the belts in a machine-shop could not be used to transmit the motion of one shaft to another; many mechanical arrangements depend for their efficiency upon it. Even so simple a matter as walking would be impossible but for the hold on the ground given to the feet by friction.

86. Kinds of Friction. An important distinction is to be made between *sliding* and *rolling* friction. The former exists where there is simply sliding motion, as in the case of a sled, or of an axle in its bearing. The latter exists where motion is accomplished through the intervention of a wheel or roller.

Sliding friction is that which is especially investigated here. The resistance due to it is much greater than that caused by rolling friction. This is seen by the effect produced when a carriage-wheel, by a shoe or some other contrivance, is made to slide instead of roll on the ground. The advantage of a wheel consists, as respects friction, in this, that instead of sliding friction on the ground, the rolling friction there and the sliding friction on the axle are substituted; the latter element is of comparatively small moment. Similarly, when heavy weights are to be moved for small distances, rollers of iron or wood are often placed under them.

87. Fluid Friction. Friction, or resistance to motion, is also felt in the case of a liquid or gas, and is then called *fluid friction*. The resistance exists both between the molecules of the fluid itself and between them and the walls of the containing vessel. This is true, for example, where water passes through pipes. This portion of the subject does not fall within the scope of this

work, but the fact must be noted with reference to a subsequent article (104).

88. Laws of Friction. Experiments upon friction have established the following laws, which hold good for any two given surfaces in contact. It is supposed that no abrasion of these surfaces takes place.

(1) The force of friction is proportional to the normal pressure of the surfaces in contact.

(2) Friction is independent of the extent of surface in contact when the normal pressure remains the same.

(3) Friction is independent of the velocity of the motion when sliding friction is considered.

89. (*a*) The normal, *i.e.* perpendicular, pressure (R), mentioned in the first law, is equal to the weight of the

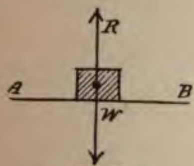

FIG. 33.

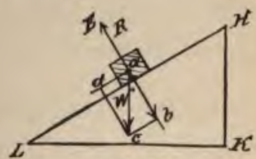

FIG. 34.

body if it rests upon a horizontal plane. Hence (Fig. 33)

$$R = W.$$

(*b*) If the body rests upon an inclined plane, the normal pressure is equal to that portion or *component* of the weight which is perpendicular to the surface. That is (Fig. 34),

$$R = W \cos \alpha.$$

For if (Fig. 34) *ac* represents the weight of the body, then *ad* and *ab* are the two partial forces, or *compo-*

nents, respectively parallel and perpendicular to the plane, into which it may be resolved (analogous to the resolution of velocities, Art. 38, p. 28, or see Art. 138, p. 144). Of these components, since $bac = HLK = \alpha$, $\therefore ad = W \sin \alpha$, and this represents the force tending to make the body slide down the plane. Also, ab, i.e. $W \cos \alpha$, is the pressure, normal to the plane, or the reaction of the plane as stated above.

(*c*) If a force P acts on the body, at an angle β, tending to make it slide along, then the normal pressure is increased (or diminished) by that component of the force acting at right angles to the plane. If (Fig. 35)

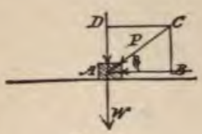

FIG. 35.

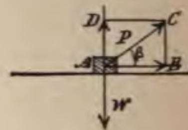

FIG. 36.

$P = CA$, exerted as a push, then $DA = P \sin \beta$, and

$$R = W + P \sin \beta.$$

If (Fig. 36) $P = AC$, exerted as a pull, then $AD = P \sin \beta$, and

$$R = W - P \sin \beta.$$

In either case, in order that the body shall be just at the point of moving, BA (Fig. 35) or AB (Fig. 36), i.e. $P \cos \beta$, must be just equal to the opposing force of friction.

90. Explanation of the Laws of Friction. (1) The FIRST LAW simply states that as this normal pressure (defined in *a, b, c* of the preceding article) is increased or

diminished, the resistance of friction increases or diminishes in the same ratio. This law can be demonstrated experimentally by an arrangement like that in Fig. 37. The weight W rests on a horizontal plane (so that $R = W$ as in a). A thread fastened to W passes over a pulley a, and is attached at the other end to a second weight P. In order that W shall be on the point of moving, P must be just equal to the force of friction (F). If W be doubled it will be found that, to satisfy the above condition, P must also be doubled, and so on. In other words, the ratio of P ($= F$) to W ($= R$) will remain constant; that is, the force of friction is proportional to the normal pressure.

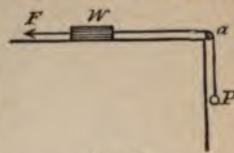

FIG. 37.

(2) The SECOND LAW may be illustrated by the case of a brick: supposing all the surfaces are alike in roughness, the friction is found to be the same upon whichever of the three surfaces it rests.

(3) The THIRD LAW is generally but not rigidly true. It is found (1) that in the case of sliding friction the resistance to motion is a little greater when the body is just about to move, and (2) that in the case of very high velocities the friction becomes sensibly diminished.

91. Coefficient of Friction. *The constant ratio between the force of friction for two given surfaces and the normal pressure is called their coefficient of friction.* If the coefficient of friction be represented by μ, then

$$\mu = \frac{F}{R},$$

and

$$F = \mu R.$$

This relation becomes on a horizontal plane

$$\mu = \frac{F}{W}, \quad \text{and} \quad F = \mu W;$$

on an inclined plane (89, *b*),

$$\mu = \frac{F}{W \cos \alpha}, \quad \text{and} \quad F = \mu W \cos \alpha.$$

The coefficient of friction is a constant relation in a given case, depending only upon the nature of the surfaces in contact.

This use of the term *coefficient*, to denote a constant factor whose value depends upon the substance involved, is a very common one in physical science. We speak, thus, of the coefficient of elasticity of a certain kind of steel; the coefficient of expansion, etc.

92. Angle of Friction. *The tangent of the angle of friction is equal to the coefficient of friction,* $\mu = \tan \alpha$.

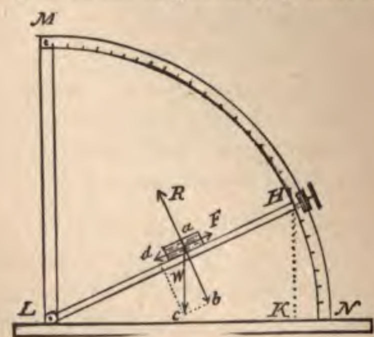

FIG. 38.

Let W be the weight of a body (Fig. 38) resting on a plane HL, and suppose that the plane is inclined at such an angle (α) that the body is on the point of sliding.

This angle is called the *angle of friction,* or *angle of repose.*

The force of friction (F), acting in the direction aF, must be equal to the component of the weight, viz. ad ($= W \sin \alpha$, as explained in Art. 89, b), which urges the body down the plane; that is,

$$F = W \sin \alpha. \tag{1}$$

Also, the normal pressure, or the reaction of the plane aR, is equal to ab, the portion of the weight acting perpendicular to LH; but $ab = W \cos \alpha$ (89, b). Hence

$$R = W \cos \alpha. \tag{2}$$

Therefore

$$\frac{F}{R} = \frac{W \sin \alpha}{W \cos \alpha} = \tan \alpha;$$

and since $\mu = \frac{F}{R}$,

$$\therefore \mu = \tan \alpha.$$

93. Determination of the Coefficient of Friction. The preceding article gives an accurate and simple method of obtaining the value of the coefficient of friction by experiment. If the surface of the plane is made of one of the materials in question, and that of the movable body in contact with it of the other, it is only necessary to observe (Fig. 38) the angle ($\alpha = HLN$) at which the plane must be elevated in order to put the body on the point of sliding; then (92) $\tan \alpha = \mu$.

The method illustrated by Fig. 37 (p. 95) may also be made use of. For, supposing the friction of the pulley to be so small that it can be neglected, if the weight W, representing one of the surfaces, is known, and also that

of P sufficient to put W on the point of moving on the other required surface, then the constant value of the ratio of these quantities is equal to the required coefficient $\left(\frac{P}{W} = \frac{F}{R} = \mu\right)$.

94. Examples of the Coefficient of Friction. The limiting values of the angle of friction for different groups of substances, as determined by experiment in the case of sliding friction, are illustrated by Fig. 39.

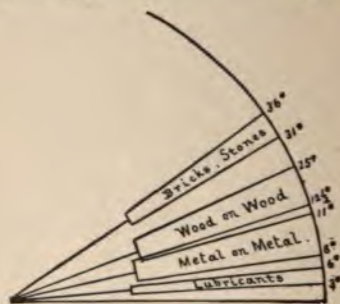

Fig. 39.

The corresponding range in the values of the coefficient of friction is, as follows

	μ	α (Angle of Friction.)
Bricks, stones........	0.60 — 0.73	31° — 36°
Wood on wood.	0.19 — 0.47	11° — 25°
Metal on metal	0.14 — 0.22	8° — 12½°
Lubricants...........	0.05 — 0.11	3° — 6°

Many different circumstances affect these values obtained by experiment, so that the above are only to be taken as average results.

For rolling friction the angle is much smaller. For example, it is stated that a railroad train in good order on a good road is not

safe against starting under the action of gravity unless the gradient is less than 18 to 20 feet to the mile ($= 0° 13'$); and that, if once started, the train will continue in motion on gradients as low as 13 feet per mile (Thurston).

EXAMPLES.

XVI. *Friction.* Articles 82–94.

1. A force of 12 lbs. is just sufficient to move a body weigh 48 lbs. uniformly along a horizontal plane: What is the coefficient of friction?

2. The value of μ is .3, the weight of the body is 16 lbs.: What force is required to move it uniformly?

3. It is found that a force of 7 lbs. suffices to move a body uniformly on a horizontal surface, where the value of the coefficient of friction is known to be .25: What is the weight of the body?

4. A body weighing 15 lbs. is just on the point of sliding when the surface it rests upon is inclined 20°: (*a*) What is the coefficient of friction and the force of friction? (*b*) If the weight of the body is doubled, what values have these quantities?

5. A body weighing 12 lbs. rests on an inclined plane whose angle of inclination is 14° and where $\mu = .4$: What is the force of friction?

6. The ratio of the dimensions of an inclined plane are as 13 (length) to 5 (height) to 12 (base): Will a body slide if the coefficient of friction is (*a*) .4 and (*b*) .5? (*c*) What is the force of friction in each case, the weight being 26 lbs.?

7. The dimensions of the plane are as in example 6: What must be the value of the coefficient of friction if the force of friction on the plane is one half the weight of the body?

8. A body weighing 12 lbs. rests on a plane where the coefficient of friction is .5: What is the force of friction (*a*) if the plane is horizontal? (*b*) if inclined 20° to the horizon?

9. The length, height, and base of a plane are 30, 18, and 24 feet: (*a*) What force is required to keep a body weighing 20 lbs. from sliding down, if $\mu = .2$? (*b*) What force is needed to draw it uniformly up the plane?

10. A body weighing 50 lbs. rests on a horizontal plane, where $\mu = .2$. What force is required to move the body uniformly if it acts as a *pull* at an angle of 30° with the plane (Fig. 36)?

11. What is the force required to move the body in example 10, if it acts at the same angle but as a *push* (Fig. 35)?

12. A force of 60 lbs. acting as a *pull* at an angle of 20° moves a body uniformly on a horizontal plane, where $\mu = .3$: What is the weight of the body?

13. If the conditions in example 12 are fulfilled when the force acts at the same angle as a push, what is the weight of the body?

14. A body weighing 4 lbs. is held against a rough vertical wall ($\mu = .6$) by a force acting at right angles to the wall: What is the force?

15. A force of 120 lbs. is just sufficient to support a body against a rough vertical wall ($\mu = .1$); the force acts at right angles to the wall: What is the weight?

[In the above examples no distinction is made between the resistance of friction when the body is just on the point of moving and that which exists when the body is already in uniform motion; in fact the former is sensibly greater than the latter.]

CHAPTER V.—WORK AND ENERGY.

A. MECHANICAL WORK—MEASUREMENT OF WORK.

95. Definition of Work. Work, in the sense in which the word is employed in Mechanics, is said to be done when a force acts upon a body and motion results in the direction of its action.

There are two essential elements here: (1) the force acting to overcome a resistance, and (2) the motion produced. Where no motion results from the action of a force, no work is done; for example, a column supporting a weight does no work.

96. Examples of Work. Work is done against gravity when a man lifts a weight up from the ground, or when he ascends, *i.e.* lifts himself up, a hill; against friction when a horse draws a carriage along a horizontal road; against the molecular force of elasticity when a spring is wound up or a bow is stretched.

97. Measurement of Work. The UNIT OF WORK is the work done by a unit force acting through the unit of distance one foot. If the force be expressed in gravitation measure (72), then the unit of work is the *foot-pound.*

(1) The work done by a constant force P, acting through a distance s, is equal to the product of the force into the distance; that is,

$$\text{the work done by } P = P.s. \qquad (1)$$

If the force acts obliquely, at an angle β with the direction of motion (Fig. 40), then only the *effective component* of the force, $P \cos \beta$, does the work. The work done is, therefore,

$$P \cos \beta . s. \qquad (2)$$

(2) The work done may also be measured by the effect produced; that is, when a weight W, expressed in pounds, is raised through a height h, expressed in feet, the work done is equal to the product of the weight raised into the vertical height; that is,

FIG. 40.

$$\text{the work done in raising a weight} = W.h. \qquad (3)$$

The *effective distance* only—that is, the vertical height—is considered. If one pound is raised vertically one foot, then one *foot-pound* of work is done, and this is the simplest form of the unit of work as above defined.

If the weight is an extended body, or if a number of bodies are considered together, the height to be taken is the vertical distance through which the centre of gravity (defined in Art. 159) is raised.

(3) Still, again, if a uniform resistance (R) is overcome through an effective distance d, then the work done is equal to the product of the resistance into the distance; that is,

$$\text{the work done against a resistance} = R.d. \qquad (4)$$

It is, in fact, immaterial, in the estimation of the amount of work done in any case, whether the attention be directed to the force acting, on the one hand, or the weight raised, or resistance overcome, on the other. For

in all cases it follows from the law of the Conservation of Energy, as explained later (101 *et seq.*), that the two must be equal; that is,

$$P.s = W.h, \tag{5}$$

or

$$P.s = R.d.$$

Whenever a weight is raised, the simplest method of estimating the amount of work done is by the product $W.h$. In some cases, however, it is more simple to measure the force acting and the distance through which it acts, and to obtain the number of foot-pounds of work done from the product $P.s$.

98. When the distance through which P acts and W is raised differ, as when, for example, by means of a set of pulleys a small power acting through a great distance raises a large weight through a small height, the equation above (5) shows that the ratio of $\frac{P}{W}$ is the inverse ratio of the distances through which they act, or $\frac{P}{W} = \frac{h}{s}$; that is—

The Power is to the Weight as the height through which the Weight is raised is to the distance through which the Power acts. This principle will be employed in the discussion of the various mechanical contrivances or machines, the lever, wheel and axle, etc., in Chapter VIII.

99. Rate of Work. The rate of work is measured by the amount of work done, for example by a steam-engine, in a unit of time. The ordinary unit employed is called the *horse-power*, and is equal to 550 foot-pounds per second, or 33,000 foot-pounds per minute. This unit is employed in measuring the efficiency of a steam-engine.

100. Applications of the above Principles. Suppose a constant force (P) acts to draw a body along a rough horizontal plane for a distance s. The work done is then equal to $P \times s$ [97 (1)]. But as it is difficult to determine the value of P directly, the principle may be made use of that: If the body moves *uniformly*, this force must be just equal to the force of friction; that is, $P = F$. But if the weight of the body is W and the coefficient of friction μ is known, then $F = \mu R = \mu W$ (91), and

$$\text{the work done against friction} = \mu W.s.$$

Suppose a weight W (Fig. 41) is raised to the top of

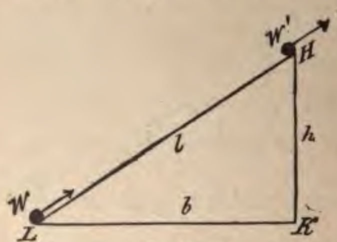

FIG 41.

an inclined plane whose height is h, and whose angle of inclination HLK is α; then (97)

$$\text{the work done in raising the weight} = W.h.$$

Also, if l is the length of the plane and μ the coefficient of friction for the surfaces in contact, since $F = \mu R = \mu W \cos \alpha$ (89, 91),

$$\text{the work done against friction} = F.l = \mu W \cos \alpha.l,$$

and the total amount of work done is equal to

$$Wh + \mu W \cos \alpha.l.$$

Since $h = l \sin \alpha$, this may be put in the form

$$Wl (\sin \alpha + \mu \cos \alpha);$$

or again, since $\cos \alpha = \frac{b}{l}$,

$$Wh + \mu Wb.$$

The last expression shows that the work done in the case supposed is the same as that required to drag the body from L to K, and then to raise it vertically to H.

EXAMPLES.

XVII. *Work.* Articles 95–100.

1. A weight of 600 lbs. is raised to the top of an inclined plane whose length is 1200 feet, and the angle of inclination $= 10°$: What work is done?

2. A weight of 150 lbs. is raised to the top of a tower along a spiral path half a mile long, and which winds about it rising at a uniform angle of 15°: How much work is done on the weight?

3. If a man in walking raises his centre of gravity a distance equal to $\frac{1}{11}$ of the length of his step (as has been estimated), how much work will he do, if he weighs 150 lbs., in walking 20 miles?

4. How much work is done by an engine weighing 20 tons in running a distance of a mile on a horizontal track, if the total resistance is 120 lbs. per ton?

5. A sled weighing 1500 lbs. is dragged 10 miles on the snow, where the coefficient of friction is .075: What work is done against friction?

6. How much work is done against friction in dragging a weight of 400 lbs. a distance of 1000 yards along a horizontal plane, if the coefficient of friction is .5?

7. A weight of 250 lbs. is dragged up an inclined plane whose length is 2600 feet and the height is 1000 feet ($\mu = .3$): How much work is done?

8. A weight of 120 lbs. is drawn along a horizontal plane for a

distance of 1000 feet, and then up an inclined plane ($a = 30°$) for the same distance; the coefficient of friction is .2 for both surfaces: What work is done?

B. ENERGY—CONSERVATION AND CORRELATION OF ENERGY.

101. Definition of Energy. *Energy is the capacity of performing work.*

The first grand principle or doctrine of energy is called the CONSERVATION OF ENERGY, which states that:

The various forms of energy may be changed into one another, but the sum total remains the same; no energy is ever lost. This is a fundamental principle in all physical science, and its importance cannot be overestimated. It is, in a certain sense, a corollary from, and an extension of, the third law of motion (69). Like these laws its truth has been established by extended series of observations and physical experiments (66).

As a proper understanding of this subject is necessary for a thorough comprehension of the laws of Mechanics, and at the same time as this is impossible without a somewhat extended discussion of the subject, it is necessary at the outset to treat it broadly in its application to all Physics and not only as restricted to Mechanics.

The term *work* must, in the first place, be understood as having a wider signification than that given in the preceding portion of this chapter. Taken broadly, work is involved in the production of any physical or chemical change. For example, work is done not only when a mass of iron is raised from the ground, but also when by heat its temperature is raised; so, too, when water is changed from its liquid form into steam; when an elec-

trical current is sent through a wire, as in the telegraph; and when the atoms of carbon of the coal unite with the oxygen of the air and combustion ensues accompanied by heat and light.

102. Forms of Energy. The forms of energy are divided into the following classes:

(1) *Mechanical energy*, or the visible energy of masses of matter, including the energy due to elasticity.

(2) *Molecular energy*, including heat, light, electricity, and magnetism.

(3) *Chemical energy*, or that produced by the chemical union of unlike atoms.

The forms of energy are also divided into (*a*) *Kinetic* * energy, or energy of motion, and (*b*) *Potential* energy, or energy of position. It will be sufficient here to apply this distinction more particularly to mechanical energy, but it also belongs to the forms which come under the other heads.

103. Kinetic and Potential Energy. (1) KINETIC ENERGY, *mechanically considered, is that which belongs to bodies in virtue of their motion.* Every moving body has a certain amount of energy, or capacity of performing work, in consequence of this motion. For example, a swiftly moving cannon-ball, a running stream, the wind —all these, because of their motion, have a definite power of doing work. In fact, their motion is a result of energy expended upon them at some previous time, which they will themselves give back if their motion is arrested. This energy of motion is sometimes called *vis viva*, or "living force," and sometimes *accumulated work*.

(2) POTENTIAL ENERGY *is that which belongs to bodies*

* From the Greek word κινέω, to move.

in virtue of their position. Every body situated above the surface of the earth has a tendency to fall to it, under the action of gravity, and this determines its potential energy, or energy of position. For example, the weights of a clock, when wound up, have, because of their elevated position, a power of doing work; *e.g.*, in turning the machinery. So, too, a body of water behind a mill-dam represents a certain amount of potential energy; that is, of energy which may be expended, *e.g.* in driving a mill if the water be allowed to fall. Also, the stones and bricks in a building represent a certain amount of energy expended once in raising them, and now present only potentially because of their position. A watch-spring when wound up, and a bow when stretched, are other examples of potential energy.

In every such example the *position* of the body in question (or of its molecules), as the *velocity* in the preceding case, indicates that a certain amount of energy has been expended upon it, and this, as before stated, will be given back on a return to the original position.

The level of the sea may be made the surface of reference for convenience; strictly, a terrestrial body would have potential energy anywhere except exactly at the centre of gravity of the earth. Moreover, other levels may be taken as the zero; for example, the weights of a clock, just mentioned, may be said to have no potential energy when they have fallen to the lowest point in their course.

104. Measurement of Energy. (*a*) Inasmuch as the amount of work done—that is, in other words, energy expended—in raising a weight W through a vertical height h is equal to Wh, it is evident that:

The POTENTIAL ENERGY *of any body is equal to the product of its weight into the distance it has to fall.*

(*b*) Again: since, if a body of weight W fall through a height h, its energy of position is entirely changed into energy of motion, this last or kinetic energy will be equal to Wh. But in falling freely from rest through a distance h, a body acquires a velocity (27) such that $v^2 = 2gh$, and $h = \frac{v^2}{2g}$. Substituting this value of h, the kinetic energy becomes $\frac{Wv^2}{2g}$. But the kinetic energy of any body of weight W and velocity v must be the same as that of the body which has acquired this velocity in falling. Hence, in general, the energy of a body in motion is expressed by

$$\frac{Wv^2}{2g}.$$

Since the mass $M = \frac{W}{g}$ (68), the value above may be expressed in this form:

$$\frac{M}{2}.v^2, \text{ or—}$$

The KINETIC ENERGY *of any body is equal to the product of one half the mass into the square of the velocity.*

105. A body of weight W and moving with a velocity v will, if brought to rest, expend against the resistance an amount of work equal to $\frac{Wv^2}{2g}$. If the resistance is uniform (as is true of friction), then, by the principles explained in Art. 97 (3), we shall have

$$\frac{Wv^2}{2g} = R.d.$$

Here $\frac{Wv^2}{2g}$ is the amount of work accumulated, or stored up, in the moving body, and $R.d$ is the work done against the resistance. If R is known, then d can be calculated, and vice versa. In the case of friction on a horizontal plane, for example, $R = F = \mu W$ (91), and hence the distance a given body will slide can be easily computed.

When d is very small, as when a heavy weight descending drives a pile a short distance in the mud, then R, the *average resistance*, will be obviously very great. It is on this principle that, when a very great resistance has to be overcome, it is often most effectual to make use of the large amount of energy stored up in a moving body of considerable mass, which may be expended through a very small distance; consider the efficiency of a heavy hammer.

106. Relation of Kinetic Energy to Momentum. The distinction must be carefully made between the momentum and the kinetic energy of a moving body. Suppose the mass of the body to be M, and the velocity v; then

$$\text{the momentum} = M.v,$$

$$\text{the kinetic energy} = \frac{M}{2}.v^2.$$

From these values it is seen that for the same body the momentum is proportional to the velocity, but the kinetic energy—that is, the power of doing work—to the square of the velocity. For example, suppose two bodies to have the same mass, but let the velocity of one be 160 feet per second, while that of the other is 80 feet per second; the former will have *twice* the momentum, but

its kinetic energy will be *four times* greater. In other words, the first body can do four times as much work in coming to rest; it will ascend vertically upward against gravity four times as far; that is, 400 feet instead of 100 feet. So, too, it will penetrate four times farther into a log of wood swung as that described in Art. 70 as a ballistic pendulum, though the momentum given to the moving mass will be only twice as great.

In the relation in Art. 70, viz.,

$$MV \pm mv = (M + m)\, v',$$

it was shown that the momentum of the two bodies after impact was equal to the sum of their momenta taken separately before, they being supposed to be perfectly inelastic. This is not true of the kinetic energy. Suppose the weights of two bodies to be 96 and 64 lbs.; then $M = 3$ $(= \frac{96}{32})$ and $m = 2$ $(= \frac{64}{32})$: also, let $V = 100$ feet per second, and $v = 50$; then, by the above equation, $v' = 80$. Now the sum of the kinetic energies of M, $(\frac{1}{2}MV^2)$ and m, $(\frac{1}{2}mv^2)$ is equal to 17,500 ft.lbs., but the kinetic energy of them moving together with the new velocity v' is only 16,000 ft.lbs. This difference would be still greater if the original motions were in opposite directions; viz., 17,500 ft.lbs. and 4000 ft.lbs. respectively.

Hence there is an apparent loss of energy after impact. The equivalent of this loss is to be found in the heat produced by the blow, as explained further in articles 108 *et seq.* This relation must be brought in in order to make Newton's third law of motion, in its general application, rigidly true.

107. Transformation of Kinetic and Potential Energy. The two kinds of energy, considered in articles 103 and

104, may be transformed the one into the other, and, if no other form of energy appears, the law of the CONSERVATION OF ENERGY requires that this interchange should go on without loss. For example, suppose a ball (Fig. 42) is rigidly attached by a rod to the support C, about which it turns without friction. If its

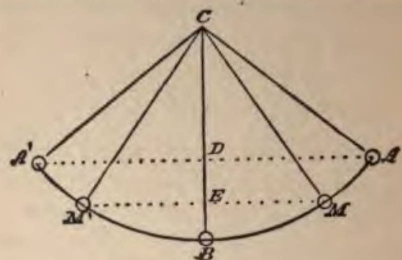

FIG. 42.

position be changed from B to A, and it be supported here for a moment, it is evident that a certain amount of work has been performed, or energy expended, upon it represented exactly by $W.h$, where h is the vertical height DB. This amount of energy has been imparted to the body, and belongs to it as potential energy in virtue of its position.

Now suppose the ball to descend; at each successive point as M, in its course from A back to B, it loses part of its potential energy, but gains a corresponding amount of kinetic energy, or energy of motion; when B is reached, all its energy is that of motion $\left(Wh = \frac{Wv^2}{2g}\right)$. In virtue of this motion it will ascend on the other side, exchanging at each point its energy of motion for energy of position, and at A' its velocity is 0 and its energy all potential.

Again, on descending, the exchange of energy takes place as before. If the supposed conditions of a perfectly rigid rod, moving without friction at C and meeting with no resistance of the air, could be realized, this motion would go on forever; it would be one kind of perpetual motion. [It would not be the "perpetual motion" sought for; for example, an engine which shall go on doing work forever without being supplied with fuel; this the doctrine of energy shows to be absurd and impossible.]

Another example will further illustrate the transformation of the two forms of energy. Suppose a ball to be projected from the ground vertically upward, with a velocity v; the energy at the moment of starting is all kinetic. By the laws of kinematics (44) it will, if gravity alone resists it, ascend to a height (h) so that $h = \frac{v^2}{2g}$. As it ascends its kinetic energy is continually exchanged for energy of position, and at its highest point it has only potential energy. If its weight is W, the amount of work done is Wh, but $h = \frac{v^2}{2g}$; hence the work done, which is the equivalent of the initial kinetic energy, is $\frac{Wv^2}{2g}$ (same result reached as in Art. 104, b). Further, on its descent it will continually exchange its energy of position for that of motion, and when it reaches the ground its energy is all kinetic. Suppose that the ball and the surface it comes in contact with are perfectly elastic. The result of the blow will be to compress the molecules of the body for an instant, and at this moment it is at rest and its energy is represented potentially by the new position of the molecules. In consequence of the elasticity, however, the molecules tend to regain their original position, and thus the potential energy is again transformed into kinetic, and the effect is to project the ball upward with the same velocity as before. In the ideal case supposed this exchange would go on continually, and the ball would bound forever.

108. Apparent Loss of Visible Energy. It is obvious that the conditions supposed in the preceding article cannot be realized, but that the pendulum and the bounding ball will sooner or later come to rest. In such cases there is an *apparent* loss of visible energy. Still more is there an apparent loss of energy when the motion of a train is arrested at a station, or that of a cannon-ball by a target. In such cases it was once believed that the energy was really lost, but it is now known that this is as untrue as it would be to suppose that the matter in a piece of paper is lost when it is burned. The matter is unchanged in amount, and is truly indestructible, though its form may be altered and it so become invisible to the eye. In an analogous way, in this apparent loss of visible energy, a new form of energy takes the place of that which disappears, for the energy itself is indestructible. This form of energy, produced as the equivalent of the mechanical energy which has disappeared, is heat.

109. Nature of Heat. Heat is now believed to be, not a form of matter, as once supposed, but a "mode of motion;" more particularly it is a very rapid undulatory vibration of the particles of matter making up the heated body. When heat is transmitted through a medium without raising its temperature it is said to be *radiated*, and the undulatory motion is believed to be propagated at a very great velocity by the particles of a supposed elastic fluid called the ether. Thus, the heat of a stove is said to be radiated in all directions from it; so, too, the heat of the sun is said to be radiated to the earth, and the heat received is called *radiant heat*, or radiant energy.

When, however, the heat is transmitted through a body at a comparatively slow rate, as from one end of an iron rod thrust in a furnace to the other, it is said to be *conducted,* and in this case the particles of the bar itself are believed to propagate the motion.

A hot body is one whose particles are in rapid motion; but "hot," as the word is used, is only a relative term, for this motion belongs to the molecules of all bodies of which we have any knowledge, however "cold," and the rapidity of the motion determines the degree of heat (temperature) as manifested, for example, to our senses or to a thermometer.

110. Examples of the Production of Heat from Mechanical Energy. Examples of the appearance of heat at the same time with the disappearance of visible energy are very common. Thus, a nail rubbed quickly with a file becomes "hot;" the same is true of a metallic button rubbed on a piece of cloth. A piece of iron on an anvil may be raised to a dull red heat by rapid blows from a hammer; a friction-match is ignited by the heat produced by a scratch; a cannon-ball whose motion is arrested by a target is itself, as well as the target, very much heated by the collision. In all such cases, as will be apparent from what has been said, the visible mass energy is exchanged for the molecular energy of heat: the slow motion of the mass for the very rapid motion of the molecules.

111. Definite Relation between Heat and Mechanical Energy. If it be true that the heat produced in the cases named (110) is the equivalent of the visible energy lost, it follows that there must be a *definite numerical ratio* between a certain amount of work done and the amount

of heat produced by it. This relation has been established in many different ways by different experimenters, but in all the cases the essential part of the process is the same, viz., to measure the amount of mechanical energy expended (in foot-pounds), and also to determine the amount of heat produced as its equivalent.

Heat is measured in *heat-units;* that is, the UNIT OF HEAT is that amount of heat required to raise one pound of water one degree in temperature. For physical problems the Centigrade thermometer is universally employed; but with English-speaking people the Fahrenheit thermometer is commonly used as the house thermometer. The relation of the two is evident from Fig. 43. For the Centigrade thermometer the freezing-point of water is made the zero, and the distance from it to the boiling-point is divided into 100 degrees. In the Fahrenheit thermometer the freezing-point is 32 degrees above the zero, and the boiling-point 212 degrees above. Hence 100 degrees Centigrade correspond to 180° (= 212° − 32°) Fahrenheit. To change the readings of either thermometer to those of the other, we have representing the number of degrees in the two scales by F and C respectively,

Fig. 43.

$$\tfrac{9}{5}C + 32° = F, \qquad \text{and} \qquad \tfrac{5}{9}(F - 32°) = C.$$

The method which was employed by Joule* was essentially as follows: A metal box B (Fig. 44) was taken full of water; in this was placed a paddle (Fig. 45) attached to a vertical spindle A, which could be revolved by means of a string passing over two pulleys C and D, and attached to two known weights E and F. If now

*The experiments of Dr. Joule were carried on between 1843 and 1849. He employed several different methods for determining the "mechanical equivalent of heat," but that which led to the most satisfactory results is the one here described.

these weights ($E + F = W$) are allowed to descend freely through a distance h, marked on the vertical scales, the work done is Wh; but this is expended in turning the paddle in the water, and, owing to the friction of the water (Art. 87), is all transformed into heat. Hence if the amount of water is known, and its temperature before and after the experiment, the number of foot-pounds of work required to produce one heat-unit—that is, to raise 1 lb. of water 1° C.—can be readily calculated. In the actual experiment it was necessary to make corrections for the loss of energy in the friction

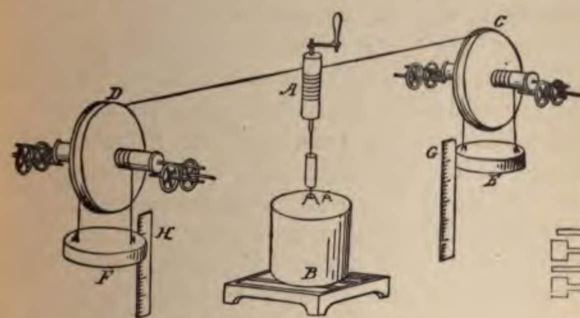

FIG. 44.

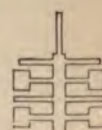

FIG. 45.

of the pulleys, the radiation of heat from the box, and several other points which need not be explained here.

The result obtained was this: *That an expenditure of* 1390 *foot-pounds of work produce one unit of heat on the Centigrade scale* (772 ft.lbs. on the Fahrenheit scale). Many other experiments have been made in various ways having as their object the determination of this same relation; for example, the amount of heat produced by the friction of two iron plates in mercury, that caused by the collision of two heavy bodies one of which has

fallen through a known height, and so on. All these experiments have confirmed the relation obtained by Joule.

112. Conversion of Heat into Work. Since a definite amount of mechanical work is equivalent to a certain amount of heat energy, the converse must also be true: that heat is convertible into mechanical work. This conversion of the former kind of energy into the other is best seen in the steam-engine. Here the burning coal in the furnace converts the water of the boiler into steam, and the expansion of this steam in the steam-chest connected with the condenser sets the piston in motion backwards and forwards. This is mechanical motion, and it may be utilized, for example, to drive the lathes in a machine-shop, to pump up water, to propel a steamship or a train of cars. All these are cases in which from heat mechanical work is obtained.

There is one most important difference between the two cases that have been described; viz., the transformation of work into heat, and that of heat into work. The former transformation can be completely made, but under no conditions, which are practicably obtainable, can a certain amount of heat be *all* changed into mechanical work. A perfect "reversible engine" (as that of Carnot) can be conceived of, but the necessary conditions cannot in practice be even approximately realized.

113. Other Forms of Molecular Energy. The form of molecular energy most closely related to Mechanics is that just considered; viz., heat. There are, however, other forms of energy into which mechanical work may be converted, and conversely from which work may be obtained. Here belong the other physical agents,

light, electricity, magnetism. The full understanding of their mutual relations requires an extended knowledge of Physics, and cannot be attempted here, but some illustrations will suffice to make the subject clear.

114. Examples of Transformation of Energy. The production of work from burning coal, already given, is one example. Another example is this: The water in a mill-pond represents a certain amount of potential energy (103); this in falling through the mill-race may be made to turn a mill-wheel, and its motion represents a certain amount of mechanical energy derived from the water; if a turbine wheel is employed, from 60 to 80 per cent of the energy of the water may be utilized under favorable conditions. The motion of the wheel may be given to a saw, which shall do work in overcoming the cohesion of the wood, or to millstones, by which grain is ground. Or, again, it may turn an electro-magnetic machine. This cannot be described here, but it is sufficient to understand that the machine accomplishes this: that the rapid mechanical motion results in the production of a current of electricity, which is the equivalent of a certain portion of the mechanical energy. This electrical energy may be conveyed by a copper wire for a considerable distance (with a loss, however, for some of it is inevitably transformed into useless heat), and then the electricity may be used to make a light, when the remainder of the energy is transformed into light and heat, or it may be used to do work in chemical separation, as in electro-plating with copper. Still, again, it may by means of a second similar machine be transformed back again into mechanical motion, and this used to do any kind of work desired.

Again, the revolution of a disc of plate-glass between

cushions suitably arranged may be made to produce electricity, which is the equivalent of part of the mechanical energy expended in turning the wheel, the remainder being expended against friction and resulting in heat. If this electricity is collected on a brass cylinder and then a spark taken from it, the light and heat of the spark, with the vibratory motion resulting in the noise, are the forms of energy into which the mechanical work has been transformed.

Chemical affinity also represents a most important kind of energy. Two dissimilar atoms under suitable conditions combine, and some other form of energy is the result. Thus, in the combustion of coal in air, it is the union of the carbon and hydrogen of the coal with the oxygen of the air which results in the formation of heat energy. So, too, the charcoal, nitre, and sulphur mixed together in gunpowder will unite under proper conditions, and the result is the appearance of energy which produces not only heat and light (and noise), but also may do a vast amount of mechanical work.

115. Conservation of Energy. To all the examples of the transformation of one form of energy into another, given in the preceding article, the law of the Conservation of Energy applies. It requires that the sum total should remain the same, that no energy should be lost in the changes. In order to prove this rigidly it would be necessary to be able to correlate all the different kinds of energy and express between them a definite ratio, as that between heat and mechanical energy. This cannot always be done, for of the real nature of some of these forms of energy but little is certainly known; but physical investigations have gone so far as to make it sure that the fundamental principle here stated is true.

116. Terrestrial Stores of Energy. From the discussion in the preceding articles it appears that, for the performance of the many kinds of work necessary for human life on the earth, the best possible use must be made of the various forms of energy which are available, for these cannot in any way be increased.

The most important stores of energy, from which mechanical work can be obtained, are the following:

1. Energy of water either potential or kinetic: this is utilized by means of the various water-wheels, and made to drive mills, etc. This includes the energy of tidal water, which is also occasionally made use of.

2. Energy of wind: employed to do work in propelling ships, and in turning windmills.

3. Energy of coal, wood, oil, and other combustibles: utilized as fuel principally in the steam-engine.

4. Energy of the muscular effort of the various animals, including man.

5. Direct energy of solar heat and light radiation: it has been found possible to employ this in running a solar engine, but thus far no extensive use has been made of it. The indirect way in which solar heat and light have been and are still being utilized is mentioned in the next article.

To the above may be added the energy of uncombined chemical elements, as sulphur and iron; also, the internal heat of the earth; the earth's rotation (note the remark at the close of the next article on tidal energy); finally, the potential energy of masses of matter above the mean surface of the earth, which have been elevated by geological changes in the past.

117. The Sun as the Ultimate Source of Terrestrial Energy. All of the important forms of energy just enu-

merated (116) are derived either directly or indirectly from the sun. The sun is constantly radiating out in all directions into space a vast amount of heat and light energy. Of the whole amount but a very minute fraction is received by the earth, and only a very small part of this is utilized, but this relatively small amount is essential to the existence of all kinds of life on the earth.

1. The heat energy of the sun causes evaporation from every sheet of water; the water thus raised in the form of vapor falls again on the earth's surface, as rain or snow, much of it at a level far above that of the sea. It forms running streams, or is collected in lakes and ponds. In descending again to the sea-level it may be made to do a great amount of work.

2. The heat energy of the sun is, also, the chief cause in setting the air in motion in the form of *winds*, and these, as have been stated, drive our ships and turn our windmills.

3. Still more important, the heat and light energy of the sun are the cause of all vegetable growth; that is, under their combined action the chemical change goes on by which the carbon (from the CO_2 in the atmosphere) is built up into the structure of the plant or tree. The energy thus appropriated is stored up, but it may be obtained again, chiefly in the form of heat, when the wood is burned as fuel. The accumulated vegetation of a former and far-distant period has been changed, though without any considerable loss of energy, to *coal*, and this therefore now represents potentially the energy of the sun received and utilized by the earth at that time. When, now, the coal is burned in the fire-box of a steam-engine, this long-stored-up potential energy be-

comes again kinetic, and from it we obtain a large part of our mechanical work.

Of the whole amount of the sun's energy which has its equivalent in the resulting vegetable growth, part, as has just been said, is obtained again in the combustion of fuel. Another part is obtained again indirectly through the muscular work of animals, who have used the vegetable growth in one form or another as food; for an animal, as regards its capacity for performing physical work, is to be regarded as a machine for the transformation of energy, which must be fed with fuel as truly as the steam-engine. A man forms no exception to this statement, for, in order that he may live and do work, he must also be fed with fuel; in his case, however, the process is one step more complex, since his principal food is the flesh of animals, who themselves have derived their support from vegetable growth.

The energy of the tides must be made an exception to the preceding remarks, for they are due to the attraction of the sun and moon; the energy of the tides may, in fact, be shown to be derived in part from the energy of the earth's rotation, the rapidity of which they consequently tend to diminish to a very small extent.

118. Dissipation of Energy. If the illustrations of the transformation of energy which have been given be carried out one step farther than is attempted, and if, too, the attempt is made to apply the principle of the Conservation of Energy to them rigidly, it will be seen that at each step in every transformation there is an apparent loss of energy (a real loss as regards useful energy) by its change into useless heat, and, moreover, that the final form which it tends to assume is always that of *heat*.

For example, a pound of coal produces upon combus-

and with the velocity thus acquired ascends another inclined plane whose dimensions are 100, 80, 60 feet respectively; the coefficient of friction is .1. How far will it go, the change of direction being supposed to take place without loss of velocity?

13. The body in example 10 is projected up an inclined plane (μ = .25) whose length, height, and base have the ratio of 10 : 6 : 8 (*a*) How far and (*b*) how long will it ascend?

14. An inclined plane has the dimensions, length 2000 feet, height 1600, base 1200, and $\mu = .2$: What velocity of projection must a body weighing 64 lbs. have just to reach the top?

15. If a body starting from rest slides down the plane in example 14, (*a*) how much work will be stored up in it when it reaches the bottom? Also, (*b*) how far will it slide on a horizontal surface ($\mu = .2$) if the change in direction occasions no loss in velocity?

[In the following examples the kinetic energy is supposed to be expended against the resistance alone; in fact, a considerable portion would result in the immediate production of heat. Moreover, the resistance is supposed to be uniform; in fact, this is not the case, and the result obtained in each problem consequently is only the *average* resistance.]

16. A hammer weighing 12 lbs. and moving with a velocity of 4 feet per second drives a nail into a plank half an inch: What resistance does it overcome?

17. A weight of 1000 lbs., used as a pile-driver, falls 20 feet, and drives the pile in one inch. What resistance does it overcome?

18. Two balls weighing 100 lbs. each are attached to the ends of a horizontal bar, this is attached to a screw of rapid pitch (236). They are made to rotate rapidly and have a velocity of 10 feet per second, when the end of the screw strikes the metal to be stamped. Suppose that the punch comes to rest after moving through $\frac{1}{10}$ inch: What resistance is overcome?

CHAPTER VI.—STATICS.

Introductory.

119. STATICS is that branch of Mechanics which considers the action of forces in so far as the body acted upon is held by them in equilibrium (61).

120. Geometrical Representation of a Force. A force may be represented geometrically by a straight line in a manner analogous to the graphic representation of velocity (20). In each case (1) the point of application, (2) the direction, and (3) the magnitude of the force are supposed to be known.

Thus, Fig. 46, (1) the position of the particle A acted upon represents the *point of application* of the force; (2) the direction of the line AB represents the *direction* of the force—that is, the direction in which it tends to move the particle A; (3) the length of the line AB represents the *magnitude* of the force, being taken proportional to this magnitude in terms of an adopted unit. It is obvious that the length of the line has meaning only when the unit of comparison is known, or when two or more forces are represented by lines in terms of the same unit; in the latter case the ratio of the lines in length will be also the ratio of the forces in magnitude.

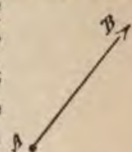

FIG. 46.

In statics a force is usually exerted as tension or pressure, and is measured in *pounds;* that is, in accordance

be equilibrium the first force must be, as stated, equal and opposite to this resultant. A body in equilibrium must be acted upon by at least *two* forces; a single force always causes accelerated motion (60).

126. Composition of Forces having the same Line of Action. (*a*) The resultant of two forces, or of any number of forces, acting in the same line and in the same direction is equal to their sum. For example, if P, Q, S, T, etc., represent forces having the same direction, as, *e.g.*, those exerted by several men pulling in the same line on a rope, and R the resultant, or equivalent single force, then

$$R = P + Q + S + T + \text{etc.}$$

Again, (*b*) the resultant of several forces acting in the same line, but some in one and others in the opposite direction, is equal to the sum of the first subtracted from the sum of the second; and it will act in the direction of the greater sum. If the respective directions of the forces be distinguished by their algebraic signs ($+$ or $-$), then the resultant will be equal to the *algebraic sum* of all the forces, and its direction will be indicated by its sign.

127. Condition of Equilibrium for Forces having the same Line of Action. The condition of equilibrium for two or more forces having the same line of action is this: *their algebraic sum must be equal to zero.*

128. Composition of two Forces not having the same Line of Action: Parallelogram of Forces. *If two forces acting on a particle be represented in direction and magnitude by the two adjacent sides of a parallelogram, then the diagonal of this parallelogram passing*

through their point of intersection will represent the magnitude and direction of the resultant.

This proposition is at once seen to be closely similar to the Parallelogram of Velocities (33 and 39). That principle, as has been stated (68, *b*), is a deduction from the second law of motion, and the same is true of the Parallelogram of Forces. The part of that law upon which both propositions are based may be stated in this form: *When several forces act simultaneously upon a body, each force produces exactly the same effect which it would have produced if it had acted singly.* This principle is true whether the body was originally at rest or in motion, and it extends as well to the case where the forces balance one another, so that the body is in equilibrium.

In applying the principle we may consider the forces either (*a*) *dynamically*, as producing motion, or (*b*) *statically*, as causing pressure or tension without motion.

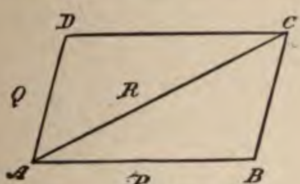

FIG. 47.

In the first case (*a*), suppose two forces P and Q to act simultaneously upon the same particle; each will have the same effect as if it acted alone, and (71) it is measured by the velocity it gives in a certain time, and its direction is that of this velocity; therefore if these velocities are represented (Fig. 47) by AB, AD respectively, then the same lines must be proportional to the forces P and Q; further, since AC represents the resultant

velocity in direction and magnitude, a force having this direction and proportional to AC must be the resultant force equivalent to the combined effects of P and Q. Hence the geometrical methods of compounding forces are the same as those of compounding velocities.

In the second case (b), the forces are generally measured by the weights they can support (72), but the same geometrical methods are also applicable to them. An experimental proof of this latter case is given in the next article.

129. Experimental Verification of the Parallelogram of Forces. Let (Fig. 48) A and B be two pulleys,

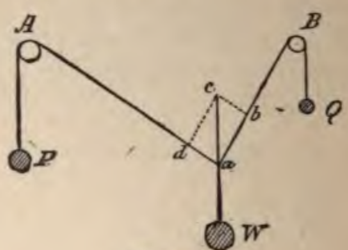

FIG. 48.

whose position may be changed at will; over these are stretched two silk threads, knotted at a, sliding without friction. At the extremities of these threads are hung two weights P and Q, and from a is hung a third weight, found by trial to be just sufficient to balance P and Q for the given position of A and B. The particle a is now in equilibrium under the action of the three forces P, Q, and W, and it is obvious that the resultant of P and Q is equal and exactly opposite to W (125).

If now, from a, ab be measured off, containing as many units of length as there are units of weight in Q, and also ad, containing as many units of length as there are units of weight in P, these forces will be represented in magnitude by ab and ad respectively, for the pulleys change their directions only. Complete the parallelogram $abcd$; it will be found on trial that the diagonal ac, which by the proposition must represent the resultant of P and Q, is vertical—that is, directly opposed to W—and also that it contains as many units of length as there are units of weight in W; hence the proposition is true in this case.

If the positions of A and B be changed, and also the magnitudes of P and Q, and a weight W be hung at a which will hold the system in equilibrium, it will be found in every case that the proposition is verified in the same manner as above, and hence it may be accepted as always true.

130. Calculation of the Resultant. *General Case.* Let the two forces P and Q, whose directions form any

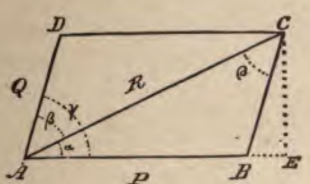

FIG. 49.

angle γ with each other, be represented (Fig. 49) by AB and AD respectively. Then, if the parallelogram $ABCD$ be completed, the line AC will represent the resultant of these forces. It is required to find an expres-

sion for the magnitude of this resultant in terms of P, Q, and γ.

From geometry (Fig. 49),

$$AC^2 = AB^2 + BC^2 + 2AB.BE, \qquad (1)$$

but $$BE = BC \cos CBE = AD \cos DAB.$$

Therefore, substituting the values of AB, AD, BE in (1), we have

$$R^2 = P^2 + Q^2 + 2PQ.\cos\gamma. \qquad (2)$$

The same formula may be shown to hold true for any other case, as when (Fig. 50) the angle γ is obtuse.

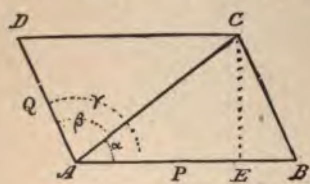

FIG. 50.

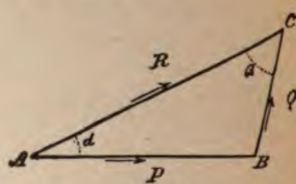

FIG. 51.

Further (Figs. 49, 50), since $BC = AD = Q$, it is seen that the relations between the two forces and their resultant are equally well expressed by the triangle ABC. This triangle is often useful for calculation, for (Figs. 49 and 51) $AB = P$, $BC = Q$, $AC = R$, $BAC = \alpha$, $ACB = DAC = \beta$, and $ABC = (180° - DAB) = (180° - \gamma)$. Therefore all the relations between the two forces and their resultant, in magnitude and direction, may be calculated from the triangle ABC by the ordinary methods of solving an oblique-angled triangle.

131. *Special Cases.* (*a*) $P = Q$. The general value of R in (2) above (130) becomes, if $P = Q$ (Fig. 52),

$$R^2 = P^2 + P^2 + 2P^2 \cos \gamma,$$
$$= 2P^2 + 2P^2 \cos \gamma,$$
$$= 2P^2 (1 + \cos \gamma).$$

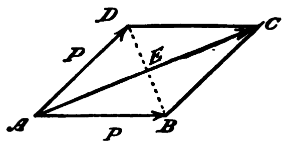

FIG. 52.

But, by trigonometry, $\cos \frac{1}{2}\gamma = \sqrt{\frac{1 + \cos \gamma}{2}}$, or $1 + \cos \gamma = 2 \cos^2 \frac{1}{2}\gamma$; therefore

$$R^2 = 2P^2 (2 \cos^2 \tfrac{1}{2}\gamma),$$
$$= 4P^2 \cos^2 \tfrac{1}{2}\gamma;$$
$$\therefore\ R = 2P \cos \tfrac{1}{2}\gamma.$$

This result, as also those of (*b*) and (*c*) below, may be obtained directly from the figures without reference to the general case. It is seen here that *the resultant of two equal forces bisects the angle between them.*

(*b*) $P = Q$, $\gamma = 60°$. The general value of R becomes for this case

$$R^2 = P^2 + P^2 + 2P^2 \cos 60°,$$
$$= 3P^2;$$
$$\therefore\ R = P\sqrt{3}.$$

(*c*) $P = Q$, $\gamma = 120°$ (Fig. 53). The general value of R becomes in this case

$$R^2 = P^2 + P^2 + 2P^2 \cos 120°,$$
$$= 2P^2 - P^2;$$
$$\therefore\ R = P.$$

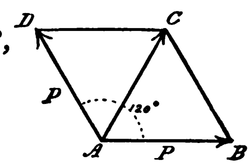

FIG. 53.

This result shows that, *if two forces are equal and inclined at an angle of* 120°, *their resultant is equal to either of them.* Also, if three equal forces acting on a particle are inclined at angles of 120° to each other, the particle will be in equilibrium.

(*d*) $\gamma = 90°$ (Fig. 54). If $\gamma = 90°$, or, in other

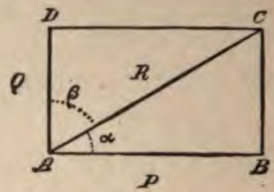

FIG. 54.

words, the two forces are at right angles to each other, the general value of R becomes

$$R^2 = P^2 + Q^2,$$
$$R = \sqrt{P^2 + Q^2}.$$

This result is derived immediately from the properties of a right-angled triangle, as are also the following relations:

$$\cos \alpha = \frac{P}{R}, \therefore P = R \cos \alpha;$$

$$\sin \alpha = \frac{Q}{R}, \therefore Q = R \sin \alpha;$$

$$\tan \alpha = \frac{Q}{P}, \therefore Q = P \tan \alpha.$$

Also, $\sin \alpha = \cos \beta$, $\cos \alpha = \sin \beta$, and $\tan \alpha = \cot \beta$.

132. Condition of Equilibrium for Three Forces acting on a Particle. *If three forces acting on a particle may*

be represented by the sides of a triangle taken in order, the particle will be in equilibrium. This proposition is called the TRIANGLE OF FORCES.

Let P, Q, S, represented by AB, AC, AD respectively (Fig. 55), be three forces acting on a particle at A,

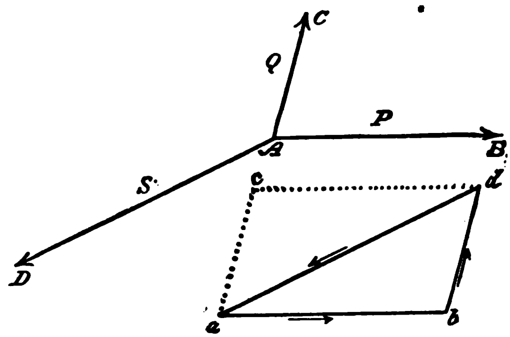

FIG. 55.

and let abd be a triangle so drawn that its sides, taken in order, represent respectively the three forces; viz., ab represents P, bd represents Q, and da represents S; then is the particle A in equilibrium.

Complete the parallelogram $abdc$. Since ac is equal and parallel to bd, the resultant of the forces ab, ac, will be the same as the resultant of P and Q; hence ad is the resultant of P and Q; but da, equal and opposite to ad, represents the third force S. Therefore, since this third force is equal and opposite to the resultant of the other two forces, the particle acted upon must be in equilibrium (125).

The condition "taken in order" is essential and must be carefully noted.

Cor. The converse of this principle is also true: that

if three forces acting on a particle keep it in equilibrium, the sides of any triangle which are respectively parallel or perpendicular to them will be proportional to these forces.

133. *If three forces acting on a particle keep it in equilibrium, each force is proportional to the sine of the angle between the other two.*

Let P, Q, S, represented by AB, AC, AD respectively (Fig. 56), be three forces acting on the particle A, and keeping it in equilibrium; then

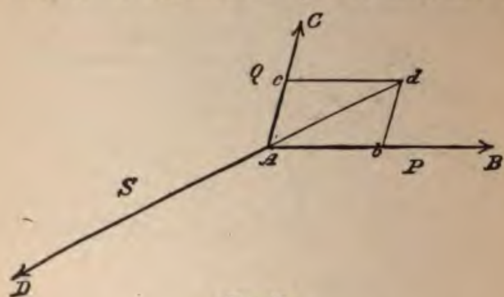

FIG. 56.

$$P : Q : S = \sin CAD : \sin BAD : \sin BAC.$$

For take Ab, Ac proportional to P and Q respectively, and complete the parallelogram $Abdc$, and draw Ad; Ad will be proportional to S and in the same straight line with it, since it has the direction of the resultant of P and Q, and is proportional to it. Then

$$P : Q : S = Ab : Ac : dA;$$

but, by trigonometry,

$$Ab : Ac \ (= bd) : dA = \sin Adb : \sin dAb : \sin dbA,$$
$$= \sin CAD : \sin BAD : \sin BAC.$$

Therefore

$$P : Q : S = \sin CAD : \sin BAD : \sin BAC.$$

Cor. The converse of this proposition is also true, and gives a second condition of equilibrium for three forces acting on a particle, which may take the place of that in Art. 132—viz.: *If of three forces acting on a particle each is proportional to the sine of the angle between the directions of the other two, the particle will be in equilibrium.* The condition must be observed, however, that no one

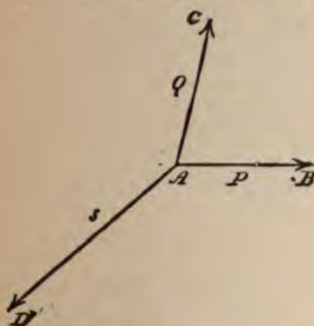

FIG. 57.

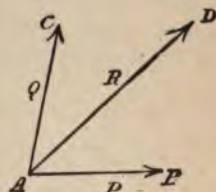

FIG. 58.

of the forces can fall between the directions of the other two; or, in other words, that the angular distance between no two of the forces can be greater than 180°. Thus the proportion may hold good for the three forces in Fig. 57, and also those in Fig. 58; but only in the former case is there equilibrium.

134. Composition of more than Two Forces acting upon a Particle. The resultant of several forces acting in the same plane upon a particle may be found (Fig. 59) by taking first the resultant of two of the forces;

then of this resultant and a third force; again, of the resultant of these three forces and a fourth force, and so on.

This method may be most simply applied as follows: Let (Fig. 60) AB, AC, AD, AE represent the four forces P, Q, S, T, acting on the particle at A. From the point a (Fig. 61) take ab in the direction of P and proportional to it; then in the same manner take bc to represent Q, cd to represent S, and de to represent T. It is obvious (compare Fig. 59) that ac represents the resultant of ab, bc, that is of P and Q; also ad of ac and cd, that is of P, Q, S; finally ae, the side which completes the polygon

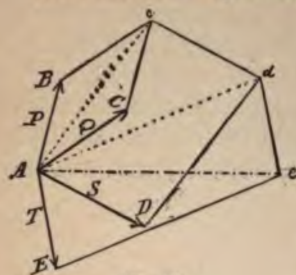

Fig. 59.

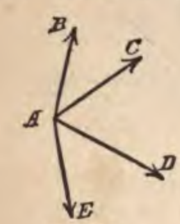

Fig. 60.

Fig. 61.

$abcde$, is the resultant of ad and de, that is of the four given forces P, Q, S, T.

The numerical calculation of the magnitude and direction of the resultant in accordance with this construction, following the method already given (130), involves considerable labor. A more simple method is given in a subsequent article (140).

135. Forces not in the same Plane. The method of finding the resultant of any number of forces acting on a particle, given in the first paragraph of the preceding article, is also applicable when the forces are not in the same plane.

For example, let AB, AC, AD (Fig. 62) represent the three forces P, Q, S respectively, acting on the particle at A. The diagonal AE of the parallelogram $ABEC$ will represent the resultant of the forces P and Q; also, if the parallelogram $AEFD$ be constructed, the diagonal AF will represent the resultant of AE and AD; that is, of P, Q, and S. The figure thus constructed is sometimes called the Parallelopiped of Forces.

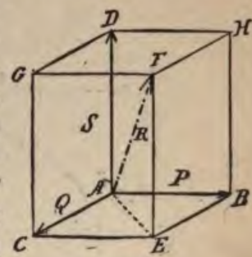

FIG. 62.

If the forces are at right angles to each other, then

$$AE^2 = AB^2 + AC^2 = P^2 + Q^2;$$

also,
$$AF^2 = AE^2 + AD^2;$$

$$\therefore R^2 = P^2 + Q^2 + S^2.$$

If α is the angle between R and P, β between R and Q, γ between R and S, then

$$\cos\alpha = \frac{P}{R}, \qquad \cos\beta = \frac{Q}{R}, \qquad \cos\gamma = \frac{S}{R}.$$

136. Condition of Equilibrium for more than Three Forces acting on a Particle. *Any number of forces acting upon a particle will hold it in equilibrium, when they may be represented by the sides of a polygon taken in order.* The forces P, Q, S, T, and U (Fig. 63),

represented by AB, AC, AD, AE, AF respectively, will hold the particle A in equilibrium if they can be represented by the sides ab, bc, cd, de, ea, *taken in order* of the polygon $abcde$. For, supposing them to be so represented, it has been shown (134) that ae is

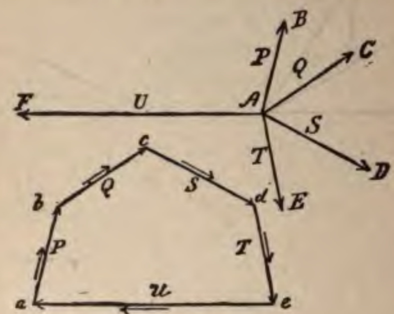

FIG. 63.

the resultant of the four forces P, Q, S, T, and since the fifth force, U, represented by ea, is equal and opposite to ae, it must hold it in equilibrium; that is, all the forces must be in equilibrium. This proposition is called the *Polygon of Forces*. It obviously applies as well to the case of forces not in the same plane.

EXAMPLES.

XIX. *Parallelogram of Forces.* Articles 126–136.

[The forces are in all cases supposed to act on a particle, or at a point of a body (123).]

1. Two forces, $P = 7$ lbs., $Q = 24$ lbs., act at right angles to each other: Required the magnitude and the direction of their resultant.

2. Two forces, $P = 13$ lbs., $Q = 7$ lbs., act at an angle of $138°$. Required the magnitude and the direction of the resultant.

3. The force $P = 16$ lbs. and the resultant $R = 24$ lbs., and

the angle between them is 42°: Required the other force, Q, and the angle between P and Q (γ).

4. The force $Q = 5$ lbs. and the resultant $R = 6$ lbs.; also, the angle between P and R (α) $= 49° \ 30'$: What is the magnitude of P and its direction?

5. Of two forces, $P = 16$ and $Q = 32$ lbs.; the angle between Q and R is 30°: Required R and the angle between P and Q.

6. A peg in a wall is pulled by two strings with forces of 8 lbs. each; they are equally inclined downward (40°) to the vertical: What weight hung on the peg would give an equal strain?

7. A peg in a wall is pulled by two strings, one horizontal with a tension of 21 lbs., and the other vertical with a tension of 28 lbs.: What single force would exert an equal pull upon it?

8. A weight is supported by two equal strings attached to nails in the ceiling and enclosing an angle of 60°; the tension of each string is 12 lbs.: What is the weight supported?

9. Two forces in the ratio of 3 : 4, acting at right angles to each other, have a resultant 25: What are the forces?

10. A boat is moored in a stream by two ropes attached to the shore making a right angle with each other; the tension of one (A) is 28 lbs., of the other (B) is 96 lbs.: (a) What is the actual force of the current, and (b) what angles do the ropes make with the direction of the current?

11. In Fig. 48, $P = 12$ oz., $Q = 15$ oz.; the angle $bad = 60°$: Required W.

12. Of two forces, $P = 2\,Q$, and they act at an angle of 45°, and $R = 16$: Find P and Q.

13. Three posts stand at the vertices of an equilateral triangle; a rope is passed completely around them, the tension of which is 24 lbs.: What is the pressure on each post?

Resolution of Forces.

137. Resolution of Forces. The process of finding the component forces whose combined effect shall be equivalent to a given single force is called the *Resolution of Forces.* It is the converse of the Composition of Forces.

To resolve a single force into two components, whose directions are given, all that is required is to construct a parallelogram on those lines having the original force as the diagonal. Thus, if (Fig. 64) $AC = R$ and the given directions are OX, OY, through C draw CD, CB parallel to the directions given; then AB, AD will be the components required. Similarly for any other directions.

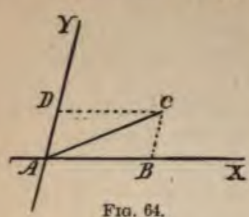

Fig. 64.

For example, let W (Fig. 65 or 66) be a weight hung by two strings knotted at a and attached at the points E and F. Produce aW vertically upward, and let ac represent the weight W, and through c draw lines parallel to aE and aF respectively; then in the parallelogram so constructed ab, ad will be the components of the force ac, which is equal and opposite to W. They give the tension of each string which supports the given weight.

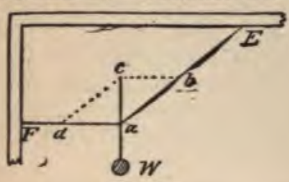

Fig. 65.

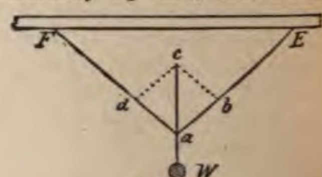

Fig. 66.

138. Rectangular Components. The case of the most importance in the resolution of a single force is that where the directions of the two components are at right angles to each other. The components in this case are (Fig. 67)

$$AB = R \cos \alpha,$$

$$AD = R \sin \alpha.$$

Here α is the angle made by the direction of the first component with that of the resultant. If the body is free to move in one of these directions only, the component in this direction is called the *effective component*, since this component alone influences the motion of the body.

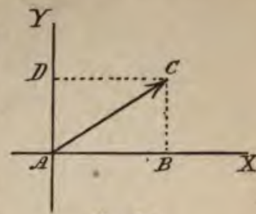

FIG. 67.

For example, suppose A to be a body either pushed along (Fig. 68) on a perfectly smooth floor as with a rod, or pulled as by a string (Fig. 69), the force in each case acting obliquely, as CA (or AC). Then the

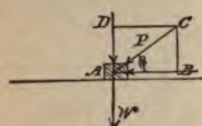

FIG. 68.

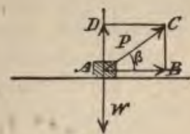

FIG. 69.

effective component, or that which alone influences the motion, is the one which acts in the direction of motion; that is, BA or AB $(=P \cos \beta)$. The other component produces no effect upon the motion. It has already been shown that if the surfaces are rough and friction has to be considered, the perpendicular component DA or AD $(= P \sin \beta)$ in the one case increases and in the other diminishes the pressure on the surface, and so alters the resistance of friction (89).

Again (Fig. 70), let a be a body resting on a smooth inclined plane; the weight (W) acts vertically downward (ac), but the body is obviously free to move only in the direction of the plane. The weight must hence be resolved along this line and along a line at right

angles to it; thus, the components of the weight are *ad* and *ab*, of which *ad* is the effective component to produce motion, and (the plane being perfectly smooth) *ab* has no influence on the motion. Now, if $HLK = bac = \alpha$, then $ad = W \sin \alpha$, and $ab = W \cos \alpha$.

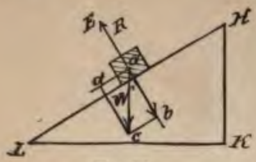

FIG. 70.

Again, let *a* (Fig. 71) be a body of weight *W* rigidly attached to the point *O*. In every position the weight acts vertically downward, but the body is free to move only in a direction perpendicular to the line of support. The two components of the weight in these directions, as indicated in each case, are then $ab = W \cos \alpha$, and $ad = W \sin \alpha$, where α is the angle made with the vertical direction.

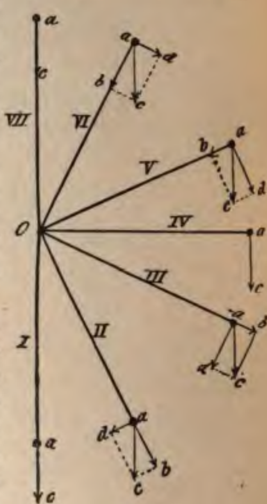

FIG. 71.

The tension of the rod, or the pull or push on the point of support, is always equal to $W \cos \alpha$ (*ab*), and the effective component to produce motion is always $W \sin \alpha$ (*ad*). Compare the different positions indicated, and note the values of the two components in each of them. In positions *I* and *VII* the moving component is zero, and the tension is equal to *W*; in position *IV* the moving component $= W$ and the tension $= 0$.

139. The explanation of the fact that a vessel may sail in a direction almost opposite to that from which the wind is blowing affords another illustration of this principle. Let *AB* (Fig. 72 or 73) represent the direction of the wind. The resultant effect upon the sail *MN* may be represented by *ab*. This force is resolved into two components, one parallel to the sail (*ac* or *db*) and pro

ducing no effect, and the other perpendicular, *ad*. But the vessel is headed in the direction *RH*; hence to find the effective component of the wind in this direction the force *ad* must be again resolved into the components *af* and *ae*. The tendency of *af* is to drift the vessel to leeward, and is nearly balanced by the resistance of the side of the vessel and keel (and the centre-board in

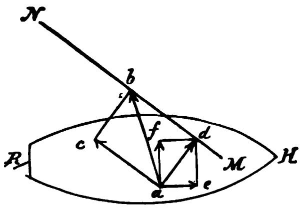

FIG. 72.

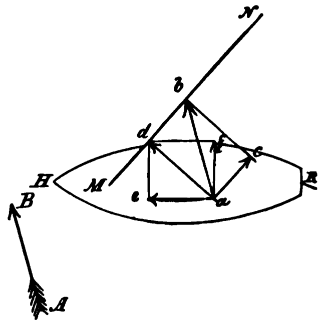

FIG. 73.

the case of a sail-boat) against the water, and the component forward is *ae*. As a matter of fact there is always a little drifting, whence the motion of the boat is kept in the required direction by the rudder. The action of the rudder is itself another example of the same principle. It is seen in the figures that with the same wind two vessels may sail in exactly opposite directions.

An explanation similar to the above may be applied to the motion of a windmill.

140. Resolution of Forces along Two Axes at Right Angles to each other. The principle of the resolution of a force along two axes at right angles to each other may be conveniently employed to obtain the resultant of a number of forces acting at a common point. Let the forces P, Q, S, T (Fig. 74) be represented by AB, AC, AD, AE, and let X and Y be any two axes at right an-

gles to each other passing through A. The components of P, Q, S, T are, geometrically,

on the axis X.... Ab, $-Ac$, $-Ad$, Ae, and
on the axis Y.... Am, An, $-Ar$, $-As$.

The minus sign indicates that the lines in question are measured—that is, that the forces act—in the oppo-

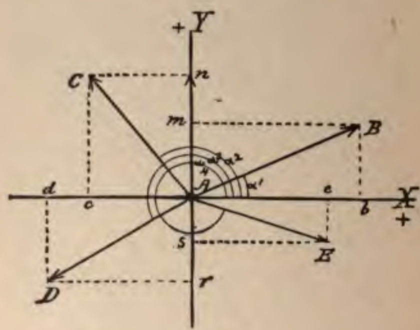

FIG. 74.

site direction to the others. The algebraic sum of each set of these components will give the components of the resultant along the respective axes.

If α', α'', α''', α^{iv} are the angles which each of the forces makes with the axis X, all measured in the same direction as indicated in the figure (74), then the two sets of components will be:

$$P \cos \alpha' + Q \cos \alpha'' + S \cos \alpha''' + T \cos \alpha^{iv} = x,$$

and

$$P \sin \alpha' + Q \sin \alpha'' + S \sin \alpha''' + T \sin \alpha^{iv} = y.$$

The directions of x and y will be indicated by the algebraic signs belonging to the sum of the components in

each case. If now the values of x and y be laid off from the point A (Fig. 75), we shall have, by completing the parallelogram, the resultant (R) represented by AF, and

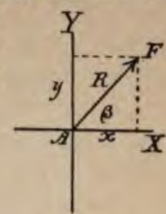

FIG. 75.

$$R = \sqrt{x^2 + y^2},$$

and

$$\tan \beta = \frac{y}{x};$$

so that the magnitude and direction of the resultant are determined.

141. Condition of Equilibrium for Three or more Forces acting on a Particle. *Any number of forces acting on a particle in the same plane are in equilibrium when the algebraic sums of their components along any two axes at right angles to each other are equal to zero.* For then $R = 0$, and this can only be true when

$$x = 0 \quad \text{and} \quad y = 0;$$

but x is the algebraic sum of the components along one axis X, and y along the other axis Y.

This condition of equilibrium (analytical condition, it is called) may be taken in place of that given in Art. 136.

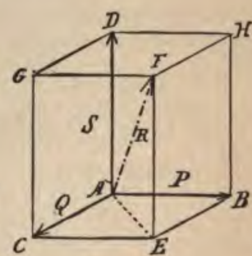

FIG. 76.

142. Resolution of a Force along Three Axes. A force may be resolved into components along any three axes not in the same plane. For example, let AF (Fig. 76) be the given force, and let the three lines drawn through A represent

the given axes. Then by reversing the construction of Art. 135 the figure represented in Fig. 76 is completed, in which AB, AC, AD are the required components.

In the application of this method the axes are ordinarily taken at right angles to each other. Thus, if α, β, γ are the angles between the given force (R) and the axes X (AB), Y (AC), and Z (AD) respectively, the three components x, y, z are

$$x = R \cos \alpha, \qquad y = R \cos \beta, \qquad z = R \cos \gamma.$$

The resultant of any number of forces, R_1, R_2, R_3, etc., acting in different planes at the same point, may be obtained by carrying out this method; for, take α_1, β_1, γ_1 to represent the angles made by R_1 with the three axes X, Y, Z respectively, and α_2, β_2, γ_2 for the angles of the force R_2 with the same axes, and so on; also, let x, y, z represent the *sum* of the components of the forces along the respective axes; then

$$x = R_1 \cos \alpha_1 + R_2 \cos \alpha_2 + R_3 \cos \alpha_3 + \text{etc.},$$
$$y = R_1 \cos \beta_1 + R_2 \cos \beta_2 + R_3 \cos \beta_3 + \text{etc.},$$
$$z = R_1 \cos \gamma_1 + R_2 \cos \gamma_2 + R_3 \cos \gamma_3 + \text{etc.},$$

and

$$\text{Resultant} = \sqrt{x^2 + y^2 + z^2}.$$

Also, any number of forces acting in different planes on a particle will keep it in equilibrium if the sum of their components along any three axes at right angles to each other is equal to zero; then

$$x = 0, \qquad y = 0, \qquad z = 0.$$

EXAMPLES.

XX. *Resolution of Forces.* Articles 137, 138.

1. A force of 150 lbs. is exerted in a due north-east direction: What portion of it is felt north? What portion east?

2. A weight of 10 lbs. (Fig. 66, p. 144) is supported by two strings of equal length attached to nails in the ceiling: What is the tension of each of the strings for the following angles between them: 0° (parallel), 30°, 60°, 90°, 120°, 150°, 180°?

3. A weight of 20 lbs. is supported by two strings at an angle of 140°; one (a) goes (Fig. 65, p. 144) horizontally to the vertical wall, and the other (b) to the ceiling: What is the tension of the two strings?

4. If the angle in example 3 is 150°, what are the tensions of a and b?

5. A picture, whose weight is 60 lbs., is supported by a cord attached to the upper corners and carried over a nail so as to include an angle of 80°. If the top of the picture is horizontal, what are the tensions of the strings?

6. A horse drags a sled by a rope inclined at an angle of 15° with the ground; the tension of the rope is 600 lbs.: What is the effective component of the force exerted? What becomes of the other component?

7. A weight of 18 lbs. is supported by two strings, one of which makes an angle of 30° with the vertical, and the other 60°: Find the tension of each string.

XXI. *Resolution of Forces along Two Rectangular Axes.*
Articles 140, 141

1. Find the magnitude and direction of the resultant of the following forces: $P = 100$ lbs., $Q = 50$, $S = 200$; the angle between P and $Q = 60°$, between Q and $S = 120°$.

2. Required the magnitude and direction of the resultant of the following forces: $P = Q = S = T = 100$ lbs. The angles are as follows: between P and $Q = 30°$, between Q and $S = 120°$, between S and $T = 30°$.

3. Required the magnitude and direction of the resultant of the following forces: $P = Q = 100$ lbs., $S = T = 200$ lbs. The angles are: between P and $Q = 90°$, Q and $S = 135°$, S and $T = 90$.

4. Four forces, P, Q, S, T, each equal in magnitude to 100 lbs., have the following directions: $P =$ N. 30° E., $Q =$ N. 30° W., $S =$ S. 60° W., $T =$ S. 60° E. What is their resultant in direction and magnitude?

5. Three forces, P, Q, S, each equal in magnitude to 200 lbs., act respectively N. 45° E., and S. 45° E., and S.: What is the direction and magnitude of their resultant?

6. Three forces, $P = Q = S = 100$ lbs., act respectively E., N. 30° W., S. 30° W.: What force will hold them in equilibrium?

7. What force will balance the action of the four forces $P = Q = S = T = 100$ lbs., acting respectively N. 20° E., N. 70° W., S. 45° W., S. 45° E.?

Composition and Resolution of Parallel Forces.

143. Parallel Forces. Forces are said to be PARALLEL when their lines of action are parallel. They are *like* parallel forces if they act in the same direction, and *unlike* when acting in opposite directions.

144. (1) LIKE PARALLEL FORCES. *The resultant of two like parallel forces acting on a rigid body is equal to their sum, acts in the same direction with them and at a point which divides the distance between them in the inverse ratio of the forces.*

Let (Fig. 77) the two like parallel forces P and Q act at the points A and B, supposed to be rigidly connected,

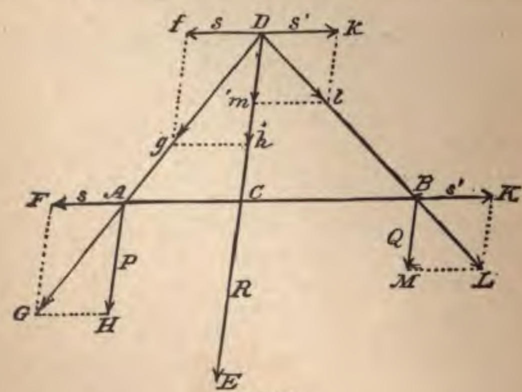

FIG. 77.

and let them be represented by AH and BM respectively. Also, at A and B let two equal and opposite forces, s and s', be applied; they will not change the con-

ditions, since they, taken alone, balance one another. Find the resultant AG of P and s, also the resultant BL of Q and s', and produce their lines of action till they meet at D. We may, by Art. 122, suppose them to act at D in their respective directions. Now resolve them (137) into their components again in directions parallel to their original directions: the components s (Df) and s' (Dk) will balance each other and may be disregarded; and the other components, P (Dh) and Q (Dm), both act in the line DC. Their resultant is therefore equal to their sum ($R = P + Q$), and may be regarded as acting at the point C.

Again, since the triangles DCA, AHG are similar, and also the triangles DCB, BML, we have

$$\frac{DC}{AC} = \frac{AH}{GH} = \frac{P}{s}, \tag{1}$$

and

$$\frac{DC}{CB} = \frac{BM}{ML} = \frac{Q}{s}. \tag{2}$$

Then dividing (1) by (2),

$$\frac{BC}{AC} = \frac{P}{Q}. \tag{3}$$

This final equation proves that the line AB is divided at C into segments which are inversely as the forces.

From (3), by inversion and composition, we obtain

$$\frac{AC}{BC + AC} = \frac{Q}{P + Q}, \quad \text{or} \quad \frac{AC}{AB} = \frac{Q}{R}. \tag{4}$$

Also, $$\frac{BC}{BC + AC} = \frac{P}{P + Q}, \quad \text{or} \quad \frac{BC}{AB} = \frac{P}{R}. \tag{5}$$

145. (2) Unlike Parallel Forces. *The resultant of two unlike parallel forces is equal to their difference, acts in the direction of the greater force, and at a point outside of it which divides the distance between the two forces externally in the inverse ratio of the forces.*

Let (Fig. 78) the two unlike parallel forces P and Q act at the points A and B, rigidly connected, and let

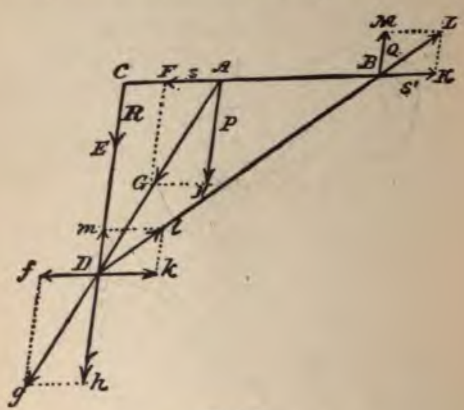

Fig. 78.

them be represented by AH and BM respectively. As before, apply two equal and opposite forces, s and s', at A and B. Find the resultant AG of P and s, and the resultant BL of Q and s'. Produce their lines of action till they meet at D; they will meet in all cases unless P and Q are equal (150). Suppose the resultants to act at this point in their respective directions, and resolve them into components parallel to the directions of P (and Q) and s. Of these components s (Df) and s' (Dk) will balance each other, and Dh $(= P)$ and Dm $(= Q)$ will act at D in the same line and in opposite directions,

Their resultant will therefore be equal to their difference (or algebraic sum); that is, in this case

$$R = P - Q.$$

Also, this resultant may be regarded as acting at C in the line CD; that is, parallel to and on the side of the greater force.

Again, since the triangles ADC and AGF are similar, as also the triangles BDC and LMB; then

$$\frac{DC}{AC} = \frac{FG}{AF} = \frac{P}{s}. \qquad (1)$$

Also,
$$\frac{DC}{BC} = \frac{BM}{ML} = \frac{Q}{s'}. \qquad (2)$$

Dividing (1) by (2), we have

$$\frac{BC}{AC} = \frac{P}{Q}. \qquad (3)$$

That is, the point C divides the line AB externally into two segments which are inversely proportional to the forces.

Also, we obtain from (3)

$$\frac{BC}{BC - AC} = \frac{P}{P - Q}, \quad \text{or} \quad \frac{BC}{AB} = \frac{P}{R}. \qquad (4)$$

Also,
$$\frac{AC}{BC - AC} = \frac{Q}{P - Q}, \quad \text{or} \quad \frac{AC}{AB} = \frac{Q}{R}. \qquad (5)$$

Taking together equations (3), (4), and (5) of the preceding article, and also the corresponding ones of this article, it is seen that: *Of two parallel forces and their resultant, each force is proportional to the distance between the other two.*

146. Experimental Verification. The principles demonstrated for like and unlike parallel forces may also be verified by experiment. (1) Let (Fig. 79) xy be a rigid rod suspended at its middle point C. Also, let two

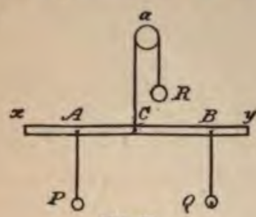

Fig. 79.

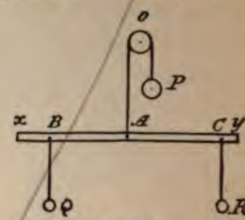

Fig. 80.

weights P and Q be taken and hung on the bar; they are then two like parallel forces. It will be found that in order to have equilibrium a weight R equal to $P+Q$ must be hung by the thread over the pulley a, and also that P and Q must be so situated that

$$\frac{P}{Q}=\frac{BC}{AC}.$$

(2) Again, let (Fig. 80) Q be hung to the rod, and P suspended by the thread over the pulley; they are then two unlike parallel forces. In this case, to maintain equilibrium R must be equal to $P-Q$. Also, Q and R must be so situated that

$$\frac{Q}{R}=\frac{AC}{BA};\text{ that is, }\frac{P}{Q}=\frac{BC}{AC}.$$

147. In the case of more than two parallel forces the resultant is found as follows: First take the resultant of two of the forces, then that of this resultant and the third force, and so on; the final resultant will be that of all the forces involved.

The point at which this final resultant of several parallel forces acts is called the *centre of parallel forces.*

148. Three Parallel Forces in Equilibrium. If three parallel forces keep a body in equilibrium, then each must be opposite to the resultant of the other two; that is, two of them must be like, and the third, equal to their sum, must act in an opposite direction at a point between them and at distances in inverse ratio to them.

149. Resolution of Parallel Forces. A single force may also be resolved into components parallel to it and to each other. To accomplish this it is only necessary to remember the rule given that the distances from the resultant force to the components are inversely as these forces.

For example, a weight W is hung at a certain point C on the rigid rod AB (Fig. 81); it is required to find

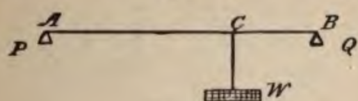

FIG. 81.

the component pressures P and Q at A and B respectively. Divide W into two such parts that $P + Q = W$, and $\frac{BC}{AC} = \frac{P}{Q}$. Or, from the final principle in Art. 145, take $\frac{AB}{BC} = \frac{W}{P}$, and $\frac{AB}{AC} = \frac{W}{Q}$.

Again, ABC (Fig. 82) is a table with a triangular top; a weight is placed at a point O; it is required to find the pressure it exerts on each of the legs at A, B, and C. Draw AD, BE, CF, each through the point O; then

$$\frac{P}{Q} = \frac{BF}{AF}, \quad \frac{P}{S} = \frac{EC}{AE}, \quad \text{and} \quad \frac{Q}{S} = \frac{DC}{DB}. \qquad (1)$$

But since the triangles AFC, FBC, as also AFO, BFO, have the same altitudes,

$$\frac{BF}{AF}=\frac{BFC}{AFC}=\frac{BFO}{AFO}=\frac{BOC}{AOC}$$

$$\therefore \frac{P}{Q}=\frac{BOC}{AOC}.$$

Similarly, $$\frac{P}{S}=\frac{BOC}{AOB},$$

and $$\frac{Q}{S}=\frac{AOC}{AOB}.$$

Therefore $$P : Q : S = BOC : AOC : AOB. \qquad (2)$$

As the problem would ordinarily be stated, the position of the point O would give immediately the segments BF, AF, and BD,

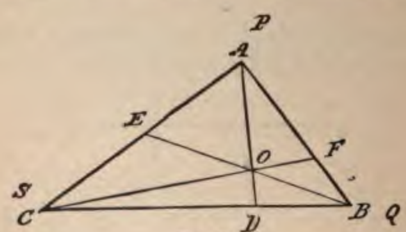

FIG. 82.

DC, etc., and therefore equation (1), remembering, also, that $P+Q+S=W$, would give the numerical solution.

If the point O is situated at the intersection of the three lines drawn from the vertices to the middle points of the opposite sides, then obviously

$$P=Q=S=\tfrac{1}{3}W. \quad \text{(See also 166, } \textit{Cor.}\text{ 1.)}$$

150. Couples. The case of two *equal* and *unlike* parallel forces is peculiar, since they have no resultant; in other words, their action cannot be balanced by the action of any single force. In Art. 145 above, if $P=Q$,

then $R = 0$; but when $R = 0$, the values of BC (4) and AC (5) become infinite. This result may also be derived directly from Fig. 78, for as the difference between P and Q continually diminishes the point C recedes, and when $P\ (AH) = Q\ (BM)$ the lines AG and BL will be parallel, and therefore C will be at an infinite distance.

Two equal and unlike parallel forces are called a COUPLE. When acting on a free body a couple tends to produce rotation, and the body can only be kept in equilibrium by the action of a second couple whose moment of rotation (as defined below) is the same and in an opposite direction.

The tendency of a couple to produce rotation is measured by the *moment* of the couple; this term will be further explained in the following article. This moment is equal to the product of either force into the distance between them. In the discussion of the general problems which arise in higher Mechanics, couples play a very important part, but in this elementary discussion of the subject reference is seldom made to them, since problems involving the rotation of bodies are for the most part excluded.

EXAMPLES.

XXII. *Parallel Forces.* Articles 143–149.

1. Find the resultant of the following parallel forces, and the position of the point at which it acts: (Compare Figs. 77 and 78.)

(*a*) $P = 5$, $Q = 7$, $AB = 48$.

(*b*) $P = 12$, $Q = 18$, $AC = 81$.

(*c*) $P = 10$, $Q = -4$, $AB = 24$.

(*d*) $P = 14$, $Q = -6$, $AC = 36$.

2 Find the force Q and the point at which it acts in the following cases·

(*a*)	$P = 3.$	$R = 8.$	$AB = 40.$
(*b*)	$P = 5,$	$R = 14,$	$BC = 15.$
(*c*)	$P = 10,$	$R = 6,$	$AB = 24.$
(*d*)	$P = 6,$	$R = 2,$	$AC = 48.$

3. A rigid rod, supported at the ends A and B, has a weight of 48 lbs hung 6 feet from A and 18 feet from B: What pressures do the supports feel? The weight of the rod itself is neglected here, as, too, in the following examples.

4. ABC is a rigid rod; at B a weight W is hung, so that $AB=12$ and $BC = 16$; the pressure at A is 32 lbs.. What is the pressure at C, and what is W?

5. $ABCD$ is a rigid rod; a weight of 4 lbs. is hung at the end A, and another of 6 lbs at C ($AC = 20$ in.); it is supported at B ($BA = 10$ in.) and D ($DA = 30$ in.): What is the pressure on the supports?

6. A weight of 144 lbs is carried by means of a rigid rod on the shoulders (at the same height) of two men A and B; the distances from them are 5 and 7 feet respectively: What weight does each carry?

7. A table has as its top an equilateral triangle ABC (Fig. 82); a weight of 20 lbs. is placed at O, so that the perpendicular distance from O on $BC = 18$ in., and those on AC, AB each equal 36 in.: What is the pressure on each of the three legs?

8. The top of a table is an isosceles triangle $AB = AC = 2\frac{1}{2}$ feet; at a point O, situated at a distance of 5 inches from each of the equal sides, a weight of 18 lbs. is placed: What pressure is felt at A, B, and C? ($BAC = 90°$.)

9. A rod, whose weight acts at its middle point, rests on two vertical props placed at the ends Where must a weight, equal to twice that of the rod, be placed that the pressure on the props shall be as 5 : 1?

10. A rod, whose weight of 18 lbs. acts at its middle point, is 4 feet long, and carries a weight of 90 lbs. 1 foot from one end: What are the pressures on two vertical props placed at the ends?

Forces tending to produce Rotation—Moments.

151. In all the cases considered thus far, the tendency of forces to produce motion of translation has alone been involved (the remarks in regard to couples are to be excepted). We have now to do with forces which tend to produce a motion of rotation.

If a body has a fixed point or axis and a force acts on it in any direction except that passing through this point or axis, it will tend to produce rotation about it. This is seen when a force acts on the edge of a wheel free to turn on an axis.

152. Moment. The moment of a force is the measure of the tendency of a force to produce rotation about a fixed point.

The moment of a force with respect to any point may be demonstrated to be equal to *the product of the force into the perpendicular distance from the point of rotation to the line of action of the force.*

This rotatory effect of the force consequently depends (1) on the magnitude of the force, and (2) on the perpendicular distance from the fixed point upon its line of action—or, briefly, upon the length of its *arm*.

For example, let (Fig. 83) a force P act at the point B on the rigid bar AB to produce rotation about the fixed point A: its moment is equal to the product of the force into its arm; viz.,

$$\text{moment of } P = P.AB.$$

This moment is increased as the magnitude of P is increased, and also as the distance AB is increased.

If the force acts obliquely, as in Figs. 84 and 85, in

this case also the product of the same factors gives the moment of the force, but the arm is now AC; that is,

$$\text{moment of } P = P.AC.$$

The same result would be obtained if the effective component of P were multiplied by the length of the

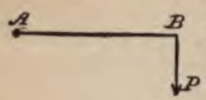

Fig. 83.

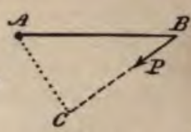

Fig. 84.

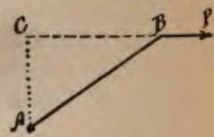

Fig. 85.

whole line AB. In the first case (Fig. 84), calling the angle $ABC = \beta$, we have, as the moment of the force,

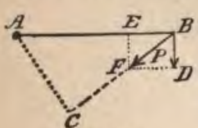

Fig. 86.

$$P.AC = P.AB \sin \beta; \qquad (1)$$

in the second case (Fig. 86), by resolving P, we have

$$P \sin \beta . AB. \qquad (2)$$

It is seen that (1) and (2) are identical. It is more convenient, and less likely to lead to error, if the rule given on page 161 in italics is uniformly observed.

153. Positive and Negative Moments. As one force may tend to turn a body in one direction, and another force in the opposite direction, it is necessary to distinguish between their moments in this particular. This is accomplished by calling the moments *positive* (+) where the tendency is to turn the body in one direction, and those *negative* (−) which have the reverse tendency.

154. Geometrical Representation of the Moment of a Force. For purposes of demonstration it is often con-

venient to consider the moment of a force as represented geometrically by double the area of a triangle having the line representing the force as its base and the given point as its vertex. Thus (Fig. 87), the moment of the force P (AB) about the point C is equal to $AB.CD$, and this is double the area of the triangle ABC.

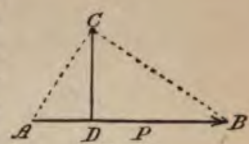

FIG. 87.

155. *The algebraic sum of the moments of two or more forces with respect to any point in their plane is equal to the moment of their resultant.*

Let AB, AD (Figs. 88, 89, 90) represent two forces, P and Q; AC, the diagonal of the parallelogram $ABCD$ constructed upon them, will be their resultant (R). The algebraic sum of the moments of AB ($= AB.Eb$) and AD ($= AD.Ed$), with respect to any point E, is equal to the moment of AC ($= AC.Ec$) with respect to the same point. There are three cases to be considered:

(a) The point E falls without the angles DAB or

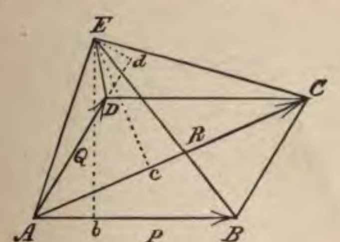

FIG. 88.

DCB (Fig. 88), and hence the moments of P, Q, and R are all of the same kind. The triangle ABE is equal to the sum of the triangles ADC and EDC, for they

have equal bases, AB and DC, and the altitude of the first triangle—that is, the perpendicular from E on AB —is equal to the sum of the altitudes of the other triangles; that is, the perpendiculars on the line DC from the vertices E and A. Hence

$$\text{triangle } ABE = \text{triangle } ADC + \text{triangle } EDC,$$
$$= \text{triangle } AEC - \text{triangle } ADE;$$
$$\therefore \text{triangle } ABE + \text{triangle } ADE = \text{triangle } AEC.$$

If we multiply this equation by 2, we have (by 154)

$$\text{moment of } P + \text{moment of } Q = \text{moment of } R.$$

(*b*) The point E falls within one of the angles named above (Fig. 89), and the moments of P and Q are of

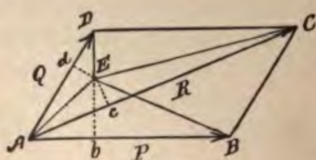

FIG. 89.

opposite kinds. The triangle AEB is equal to the difference of the triangles ADC, EDC, for they have equal bases AB and DC, and the altitude of the first triangle (the perpendicular from E on AB) is equal to the difference of the altitudes of the others (the perpendiculars from A and E on DC). Hence

$$\text{triangle } AEB = \text{triangle } ADC - \text{triangle } EDC,$$
$$= \text{triangle } AED + \text{triangle } AEC;$$
$$\therefore \text{triangle } AEB - \text{triangle } AED = \text{triangle } AEC.$$

Multiplying this equation by 2, we obtain (by 154)

$$\text{moment of } P - \text{moment of } Q = \text{moment of } R.$$

(*c*) The point E falls on the line of the resultant (Fig. 90). Since the perpendicular distances from B

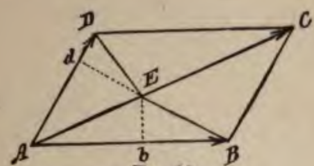

FIG. 90.

and D on AC are equal, the triangles AED, AEB have the same base and equal altitudes, and are therefore equal.

$$\text{triangle } AED = \text{triangle } AEB,$$

or
$$\text{triangle } AED - \text{triangle } AEB = 0.$$

Multiplying by 2, we have

$$\text{moment of } P - \text{moment of } Q = 0.$$

This is in accordance with the proposition, for the moment of the resultant is obviously zero for this final case. The principle here established is an important one: *the algebraic sum of the moments of two forces is zero for any point on the line of their resultant.*

The result reached in this article may be readily extended to any number of forces acting in the same plane, whether they intersect at a common point or are parallel.

156. *A body, free to turn about a fixed axis and acted upon by forces in a plane perpendicular to this axis, will*

be in equilibrium if the algebraic sum of the moments of all the forces about this axis is zero.

According to the condition the body is only free to rotate in one plane perpendicular to its axis. In order that it should be in equilibrium the tendency to rotate in one direction must be balanced by the tendency to rotate in the opposite direction. This condition is satisfied only when the algebraic sum of the moments of all the forces with respect to the axis is zero. By the concluding paragraph in the preceding article it is evident that the resultant (unless equal to zero) of all the forces must pass through this axis, for only in this case can its moment be equal to zero.

This proposition is a most important one and has many applications; it is often called the PRINCIPLE OF THE LEVER.

157. Free and Constrained Body. A body which may move unrestrained in any direction is said to be *free.* On the other hand, a body whose motion is restricted in any way is said to be *constrained.*

An example of a constrained body is mentioned in the preceding article, and the condition of equilibrium for such a body, free to rotate only, is there given. Another example would be the case of a body strung on two wires and free only to slide in their direction; that is, to have motion of translation. The obvious condition of equilibrium here is that the algebraic sum of the components of the forces taken in the given direction should be equal to zero.

The conditions of equilibrium for a free body, acted upon by any number of forces in one plane, require that (a) it should not slide—that is, have motion of translation—and that (b) it should not rotate.

For (a) the algebraic sum of the components in any two directions at right angles to each other must be equal to zero (141).

For (b) the algebraic sum of the moments of the forces about any point in the plane must also reduce to zero (156).

EXAMPLES.

XXIII. *Moments.* Articles 151-156.

1. A force, $P = 12$ lbs., acts at right angles to an arm 6 feet long: What is its moment?

2. A rigid rod AB, 8 feet long and free to turn about B, is acted on by a force, $P = 64$ lbs., whose direction makes an angle of 40° with AB: What is the moment of P?

3. A force, $P = 150$ lbs., acts at the extremity of a rod, AB, 12 feet long, and at an angle of 160°: What is the moment of P about B?

4. A bar 6 feet long and pivoted at the middle has a weight of 24 lbs. hung at one extremity: What is the moment of the weight (a) when the bar is horizontal, (b) when it makes an angle of 40° below, and (c) of 60° above with the horizontal position?

[Other examples involving the moments of forces are given under the Lever.]

Summary of Conditions of Equilibrium.

158. The various CONDITIONS OF EQUILIBRIUM for forces acting on a body in one plane, which hold true under the various circumstances, may be summed up here as follows:

(A) *For two forces:* They must (1) act at the same point; (2) they must be opposite; and (3) they must be equal.

(B) *For three forces:* They must, produced if necessary, (a) act at the same point, and have (α) the same

or (β) different lines of action; or (b) they must be parallel.

(a) α. If they act in the same line, their algebraic sum must be equal to zero (127).

β. (1) If they act at the same point and not in the same line, they may be represented by the sides of a triangle taken in order (132); or, (2) Each will be proportional to the sine of the angle between the directions of the other two (133, *Cor.*).

(b) If parallel, two must be like parallel forces, and the third must be equal to their sum and act in an opposite direction to them at a point distant from them in the inverse ratio of the forces (148).

(C). *For more than three forces:*

(a) α. If they act in the same line, their algebraic sum must be equal to zero (127).

β. If they act at the same point (produced if necessary), but not in the same line, then: (1) They may be represented by the sides of a polygon taken in order (136); or, (2) The algebraic sum of their components along any two lines at right angles to each other must be equal to zero (141).

(b) If they are parallel, the algebraic sum of their moments with respect to any point in the plane must be equal to zero.

(c) If they act at different points or in different directions, then: (1) The algebraic sum of their components along any two lines at right angles to each other must be equal to zero; and also, (2) The algebraic sum of the moments of the forces about any point in the plane must be equal to zero (157).

CHAPTER VII.—CENTRE OF GRAVITY.

A. CENTRE OF GRAVITY OF BODIES—PLANE AND SOLID.

159. Definition of the Centre of Gravity. The attraction of the earth upon all particles of matter upon its surface is exerted in the direction of lines drawn to the centre. For the particles of the same body, or of neighboring bodies, these lines may be regarded as parallel. For a given body the resultant of all these parallel forces will act, whatever its position, at a certain point, called the *centre of gravity.* Hence

The centre of gravity of a body is that point at which the whole weight of the body may be considered as concentrated; or—

It is a point at which the body, if supported there and if acted upon only by gravity, will balance in every position.

The definition may be extended to the case of a system of bodies if we suppose them and their centre of gravity to be rigidly connected.

160. The Centre of Gravity of Two Bodies. Let P and Q (Fig. 91) be any two bodies of known weight. It is required to find the position of their centre of gravity. The weights may be considered as two like parallel forces whose resultant (144) will be equal to their sum and will act at a point which

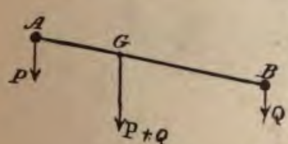

FIG. 91.

shall divide the distance between them in the inverse ratio of the forces. Therefore, if the straight line AB be drawn and the point G taken on it, so that

$$\frac{AG}{BG} = \frac{Q}{P},$$

G will be the centre of gravity of P and Q. If this point be rigidly connected with the two bodies, the system, supported there, will balance in every position.

161. The Centre of Gravity of any Number of Bodies. Let P, Q, S, and T (Fig. 92) be four bodies of known weight and occupying certain positions with reference to each other. It is required to find their common centre of gravity. On the straight line AB joining the positions of P and Q take E, so that

Fig. 92.

$$\frac{AE}{EB} = \frac{Q}{P};$$

then, by 160, E will be the centre of gravity of P and Q. Again, suppose $P + Q$ to act at E, and on the line EC take F, so that

$$\frac{EF}{FC} = \frac{S}{P + Q};$$

then F is the centre of gravity of P, Q, and S. Again, suppose $P + Q + S$ to act at F, and on the line DF take G, so that

$$\frac{FG}{DG} = \frac{T}{P + Q + S};$$

then G is the centre of gravity of the four bodies P, Q,

S, and T, and if it were rigidly connected with them the system would balance in every position.

This method could obviously be extended, whatever the number or positions of the given bodies.

162. The Centre of Gravity of a Straight Line. *The centre of gravity of a straight line is at its middle point.* Suppose the line to be made up of a series of material particles of equal weight; the centre of gravity of each pair of them taken at equal distances from the centre of the line will be at this point. Hence the centre of gravity of the whole line will be at its centre.

163. The Position of the Centre of Gravity of any Plane Figure determined by its Symmetry. *The centre of gravity of any geometrical figure, which is symmetrical with*

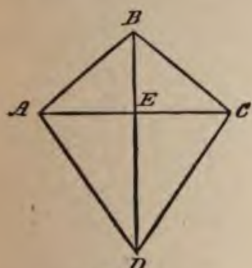

Fig. 93

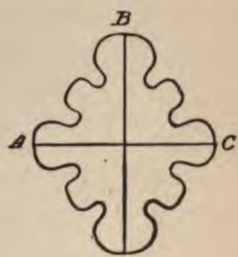

Fig. 94.

reference to an axis, lies in this axis. By a plane figure is here meant any material geometrical figure whose thickness is uniform and indefinitely small in reference to its other dimensions.

Suppose the figure to be made up of parallel material lines, all bisected, according to the supposition, by the axis of symmetry. The centre of gravity of each of

these lines (162), and, therefore, of all of them taken together—that is, of the whole figure—will lie in this axis. If the figure has two axes of symmetry, their point of intersection will determine the centre of gravity.

For example, the quadrilateral $ABCD$ in Fig. 93, made up of two isosceles triangles placed base to base, is symmetrical with reference to the axis BD, for it bisects at right angles all lines drawn parallel to AC; hence the centre of gravity of the figure is in BD. So also if, in Fig. 94, AC and BD are both axes of symmetry, the centre of gravity must lie at their point of intersection.

164. Centre of Gravity of Regular Polygons. The position of the centre of gravity of the regular polygons is given immediately by this principle of symmetry. For example, in the equilateral triangle (Fig. 95) it is at G, the intersection of the three axes of symmetry drawn

FIG. 95.

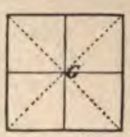

FIG. 96.

FIG. 97.

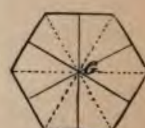

FIG. 98.

from the vertices to the middle points of the opposite sides; in the square (Fig. 96), at the intersection of the lines joining the middle points of the two opposite sides, or of the dotted lines joining the opposite angles; in the regular pentagon (Fig. 97), at the common point of intersection of the five lines, each drawn from a vertex to the middle of the side opposite; in the hexagon (Fig. 98), at the intersection of the three lines joining the middle points of the opposite sides, or of the three

dotted lines joining the opposite angles; and so on. In the circle, every diameter is an axis of symmetry; the centre of gravity is consequently at the centre.

165. Centre of Gravity of a Parallelogram. *The centre of gravity of a parallelogram is at the point of intersection of the two diagonals.* Let $ABCD$ (Fig. 99) be a parallelogram whose diagonals intersect at G; this point is the centre of gravity. Draw any line dgb parallel to the diagonal DGB; it is bisected by the other diagonal AGC. For from the similar triangles Adg, ADG, and Agb, AGB,

$$\frac{dg}{DG} = \frac{Ag}{AG}, \quad \text{and} \quad \frac{gb}{GB} = \frac{Ag}{AG};$$

$$\therefore \frac{dg}{DG} = \frac{gb}{GB}, \quad \text{or} \quad \frac{dg}{gb} = \frac{DG}{GB}.$$

But, by geometry, $DG = GB$; hence $dg = gb$; therefore the centre of gravity of this line (162) must be at g. Hence, if the whole figure be considered as made up of

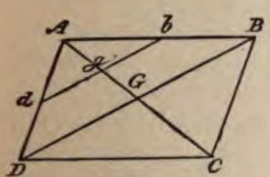

FIG. 99.

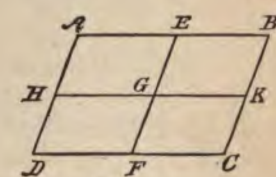

FIG. 100.

material lines parallel to DB, the centre of gravity of each—that is, of the whole figure—must be in AC. In the same way it may be shown to lie in DB, and therefore it must be at their point of intersection G.

It may also be shown that the same point is deter-

mined by the intersection of the lines (Fig. 100) EF and HK joining the middle points of the opposite sides.

166. Centre of Gravity of a Triangle. *The centre of gravity of any triangle is on the line drawn from either vertex to the middle point of the opposite side, and one third of the distance from this side.*

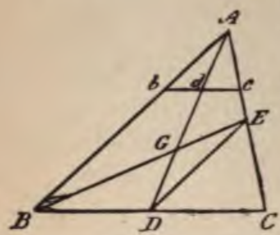

FIG. 101.

Let ABC be a triangle (Fig. 101); from the vertex A draw AD to the middle point of the opposite side BC; also draw any line bdc parallel to BDC. Since the triangles Abd, ABD, and Adc, ADC, are similar,

$$\frac{bd}{BD} = \frac{Ad}{AD}, \quad \text{and} \quad \frac{dc}{DC} = \frac{Ad}{AD};$$

$$\therefore \frac{bd}{BD} = \frac{dc}{DC}, \quad \text{or} \quad \frac{bd}{dc} = \frac{BD}{DC}.$$

But, by construction, $BD = DC$; hence $bd = dc$, and the centre of gravity of the line bc is at d, on the line AD. Therefore it follows that the centre of gravity of all the material lines parallel to BC, of which the triangle may be considered as made up, lies in AD, and hence also that of the whole figure.

Draw from the vertex B the line BE to the middle point E of the side AC; in the same way it may be proved that the centre of gravity of the triangle must lie in BE. Hence it must be at the intersection of AD and BE; that is, at G.

Connect DE; since the triangles AGB and DGE are similar, and also the triangles ABC and EDC,

$$\frac{AG}{GD}=\frac{AB}{DE}, \quad \text{and} \quad \frac{AB}{DE}=\frac{BC}{DC};$$

$$\therefore \frac{AG}{GD}=\frac{BC}{DC}=\frac{2}{1}.$$

That is, GD is one half of AG and one third of AD.

Cor. 1. The centre of gravity of three heavy bodies of equal weight will coincide with the centre of gravity of the triangle whose vertices occupy the position of the three bodies. For (Fig. 102) the centre of gravity of the equal weights B and C will be at D, so that $BD = DC$ (160). Also, the centre of gravity of B and C together at D, and of the third weight at A, will be at G,

so that $\dfrac{AG}{GD}=\dfrac{2}{1}$, or $\dfrac{AD}{GD}=\dfrac{3}{1}$.

But the same point G is also the centre of gravity of the triangle ABC.

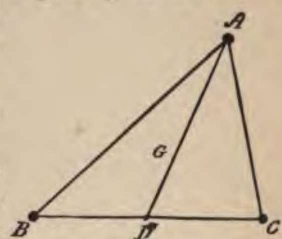

Fig. 102.

Cor. 2. The centre of gravity of a polygon can be found by dividing it into triangles, taking the centre of gravity of each by the above article and then proceeding as in Art. 161. The weights of the triangles are taken as proportional to their areas.

167. The Centre of Gravity of a Solid Figure. *The centre of gravity of a solid, which is symmetrical with reference to any plane, must lie in this plane.* This follows from the same consideration as that in Art. 163. If there are two planes of symmetry, the centre of gravi-

ty will lie in their line of intersection; and if three, at the point in which they all intersect. Therefore the centre of gravity of a sphere is at its centre; of a cylinder, at the middle point of its axis; of a rectangular solid, at the point of intersection of three planes drawn parallel to, and midway between, each pair of opposite sides.

168. The Centre of Gravity of a Triangular Pyramid. *The centre of gravity of a triangular pyramid is on the line drawn from a vertex to the centre of gravity of the opposite side, and one fourth of the distance from that side.* Let $ABCD$ (Fig. 103) be a triangular pyramid. Take E, the middle point of DC, and draw BE; then F, one third of the distance on BE from E, is (166) the

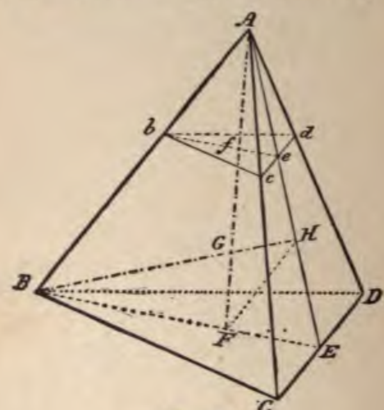

FIG. 103.

centre of gravity of the base of the pyramid. Let bcd be the triangle formed by the intersection of a plane drawn through any point b parallel to the base BCD. Draw AE meeting cd at e, and AF intersecting be at f; then it may be shown that f is the centre of gravity

of the triangle bcd. For from the similar triangles Ace, ACE, and Aed, AED,

$$\frac{ce}{CE} = \frac{Ae}{AE}, \quad \text{and} \quad \frac{ed}{ED} = \frac{Ae}{AE};$$

$$\therefore \frac{ce}{CE} = \frac{ed}{ED}, \quad \text{or} \quad \frac{ce}{ed} = \frac{CE}{ED}.$$

But $CE = ED$; hence $ce = ed$, and e is the middle point of cd. Again, from the similar triangles Abf, ABF, and Afe, AFE,

$$\frac{bf}{BF} = \frac{Af}{AF}, \quad \text{and} \quad \frac{fe}{FE} = \frac{Af}{AF};$$

$$\therefore \frac{bf}{BF} = \frac{fe}{FE}, \quad \text{or} \quad \frac{bf}{fe} = \frac{BF}{FE}.$$

But $BF = 2FE$; hence $bf = 2fe$, and $fe = \frac{1}{3}be$, and f is a point on the line drawn from the vertex b to the middle point of the opposite side one third of the distance from that side; hence (166) f is the centre of gravity of the triangle bcd.

If now the whole figure be thought of as made up of material triangles all parallel to BCD, the centre of gravity of each one, and hence of the whole pyramid, will lie in the line AF. For the same reason it will lie in the line BH, drawn from B to the centre of gravity of the side ACD; hence it will be at their point of intersection G.

Since now the triangles HGF and BGA are similar, as also the triangles HFE and ABE, we have

$$\frac{FG}{AG} = \frac{FH}{AB}, \quad \text{and} \quad \frac{FH}{AB} = \frac{FE}{BE};$$

$$\therefore \frac{FG}{AG} = \frac{FE}{BE} = \frac{1}{3}.$$

Therefore FG is one third of AG and one fourth of the whole line AF.

Cor. The centre of gravity of four heavy bodies of equal weight coincides with the centre of gravity of the triangular pyramid at whose vertices these bodies are situated. This follows in the same way as did *Cor.* 1, Art. 166.

169. To find the centre of gravity of a pyramid, having any rectilinear polygon as its base: Divide this base into triangles by lines drawn from any angular point to the others, and suppose planes passed through the vertex and these lines. The pyramid is divided into a number of triangular pyramids. The centre of gravity of each of these will lie on the line drawn from the common vertex to that of its base and one fourth of the distance from the base; therefore the centre of gravity of the whole pyramid will lie in a plane parallel to the base and one fourth the distance from it.

Again, suppose the pyramid made up of similar polygons parallel to the base; the centre of gravity of each, and therefore of the whole figure, will lie on a line drawn from the vertex to the centre of gravity of the base (determined as in Cor. 2, Art. 166). The centre of gravity of the pyramid will be at the point where this line intersects the plane above determined; that is, one fourth of the distance from the base.

170. To find the centre of gravity of a cone: Suppose the cone be divided into an infinite number of triangular pyramids; then, by the reasoning of the preceding article, it is obvious that the centre of gravity must lie in a plane parallel to the base and one fourth the distance from it to the vertex. and also in the line joining the

latter point with the centre of gravity of the base, and hence at their point of intersection.

The centre of gravity of the material surface (taken in the same sense as in Art. 163) of a right cone lies on its axis and one third the distance from the base to the vertex. This is proved by showing it to be true first for a pyramid whose sides are triangles, and then passing to the cone which is the limit of the pyramid when the sides are indefinitely increased in number.

171. Problems. (1) *Given the positions of the centres of gravity of two known parts of a body, to find the centre of gravity of the whole.* Let the weights of the parts be w' and w'' acting at the points g' and g''; then the centre of gravity of the whole will be on the line $g'g''$ at a point G so situated that

$$\frac{w'}{w''} = \frac{g''G}{g'G}.$$

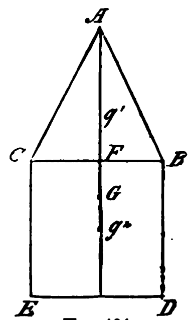

FIG. 104.

For example, suppose the parts to be the isosceles triangle ABC and the square $BDEC$, situated as in Fig. 104, and let $AF = s$ (side of square). The centre of gravity of the triangle is at g' ($g'F = \frac{1}{3}AF$), and of the square at g'' ($Fg'' = \frac{1}{2}s$); also, $\frac{w'}{w''} = \frac{1}{2}$. Therefore $\frac{w'}{w''} = \frac{1}{2} = \frac{g''G}{g'G}$; but $g'g'' = \frac{5}{6}s$, hence $g'G = \frac{5}{9}s$, and $g''G = \frac{5}{18}s$.

(2) *Given the positions of the centres of gravity of a body and of a known part, to find that of the remainder.* Let W be the weight of the whole, and w' of the part; then that of the remainder $= W - w'$ $(= w'')$; also, let

the centre of gravity of W be at G, of w' at g', and of the remainder (w'') at g''. Join g' and G, and take on the line produced $\frac{Gg''}{Gg'} = \frac{w'}{w''} = \frac{w'}{W - w'}$; then is g'' the required point.

For example, let the whole body be a circle ABC (Fig. 105) whose centre is G; and the part a second

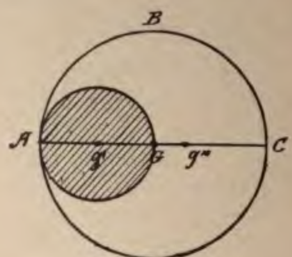

FIG. 105.

circle whose centre is g', and whose diameter is the radius (R) of the larger circle. Then $\frac{w'}{W} = \frac{1}{4}$, and $\frac{w'}{W - w'}\left(\text{or } \frac{w'}{w''}\right) = \frac{1}{3}$; also, $g'G = \frac{1}{2}R$; hence $\frac{Gg''}{Gg'} = \frac{1}{3}$, and $Gg'' = \frac{1}{6}R$.

EXAMPLES.

XXIV. *Centre of Gravity.* Articles 159–171.

[The connecting rods mentioned are supposed to be rigid, and, except when otherwise stated, to be uniform and without weight.]

1. Where is the centre of gravity of two bodies, A and B, weighing 4 and 5 lbs. respectively, rigidly connected by a weightless rod 24 inches long?

2. Three weights of 3, 6, and 12 lbs. are hung, at A, B, and C

respectively, on a rigid bar; $AB = 6$ inches and $BC = 12$ inches: Where will the bar balance?

3. Four weights of 2, 3, 5, and 6 lbs., at A, B, C, and D, are connected rigidly in a straight line; $AB = 10$, $BC = 8$, $CD = 18$: Where is their centre of gravity?

4. A rod AB, 18 inches long and weighing 4 ounces, has a weight $P = 2$ lbs. hung at the end B: Where will it balance?

5. A rod AB, 24 inches long and weighing half a pound, has a weight $P = 5$ lbs. at a point 1 inch from B: Where will it balance?

6. What weight must be hung at the end of a rod 3 feet long and weighing half a pound that it may balance 3 inches from that end?

7. A rod 2 feet long and having a weight of 5 lbs. at one end balances at a point $\frac{4}{7}$ of an inch from this end: What is its weight?

8. A heavy rod, 24 inches long and weighing 3 lbs., balances alone at a point 10 inches from one end: What weight must be hung at the other end in order that it may balance exactly in the middle?

9. A ladder 40 feet long and weighing 60 lbs. has its centre of gravity 16 feet from the larger end: (a) If supported by two men, A and B, at the extremities, what will they carry? (b) Where should A stand to divide the weight equally with B?

10. A uniform rod AB, 20 inches long and weighing 3 lbs., has a weight of 12 oz. at the end A, and one of 6 oz. two inches from B: Where will it balance?

11. Weights of 8, 4, and 18 oz. respectively are placed at the vertices A, B, C of a triangle right-angled at B; $AB = 18$ inches, $BC = 9$ inches: How far from C is their centre of gravity, and on what line?

12. ABC is an isosceles triangle; $AB = AC = 25$ inches, $BC = 14$ inches; weights of 4 lbs. each are placed at A, B, C respectively: Where is their centre of gravity, measured from A?

13. ABC is a uniform rod bent at right angles at B; $AB = BC = 12$ inches: Where is its centre of gravity, measured from B?

14. A uniform square board, $ABCD$ ($AB = 24$ inches), weighs 3 lbs.: (a) Where will it balance if a weight of 1 lb. is placed at A? (b) if equal weights of 1 lb. each at A and B? (c) if equal weights of 1 lb. each at A, B, and C?

15. Where is the centre of gravity of the remainder of a square board, $ABCD$ ($AB = 24$ inches), (a) after a piece is cut out by lines drawn from A and B to the centre? (b) Again, if a piece is cut out by lines joining the centre with the middle points of two adjacent sides? (c) Again, if one corner is cut off by a line joining the middle points of two adjacent sides?

16. ABC is an isosceles triangle; $AB = AC = 20$ inches, $BC = 32$ inches; the upper portion is cut off by a line joining the centres of these sides: Where is the centre of gravity of the remainder?

17. A circle having a diameter of 12 inches has a smaller circle cut out of it; the diameter of the latter is the radius of the former: Where is the centre of gravity of the remainder?

18. A circle has a diameter of 16 inches; a smaller circle tangent to it and having a diameter of 12 inches is cut out of it: Where is the centre of gravity of the remainder?

19. Find the centre of gravity of a frustum of a right cone whose altitude is 8 inches, and the diameters of the two bases 6 and 12 inches respectively.

20. Find the centre of gravity of a figure made up of two isosceles triangles (Fig. 93, p. 171): $BE = 6$, $ED = 12$.

21. Find the centre of gravity of a figure made up of a square and an isosceles triangle, the latter having its base coincident with and equal to a side of the square (Fig. 104); the altitude of the triangle, 12 inches, is twice the side of the square.

22. Two uniform cylinders of equal lengths (= 20 inches), and having diameters of 12 and 6 inches, are joined so that their axes coincide: Where is the centre of gravity?

B. APPLICATION OF THE PRINCIPLES OF THE CENTRE OF GRAVITY—EQUILIBRIUM AND STABILITY.

172. Condition of Equilibrium. *A body supported at a point, or on an axis, and free to turn about it, will be in equilibrium, under the action of gravity, if the vertical line through the centre of gravity passes through the point or axis of support.* Such a body (Figs. 106, 107, 108) is acted upon by two forces, (1) the weight acting

vertically downward through the centre of gravity, and (2) the reaction through the point or line of support. Therefore (158, A) the body can be in equilibriun only as these forces are equal and opposite; that is, the verti-

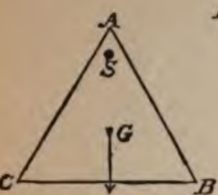

Fig. 106.

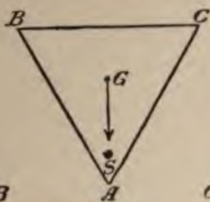

Fig. 107.

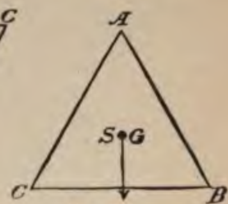

Fig. 108.

cal line through the centre of gravity must pass through the point of support.

173. Hence, to find the centre of gravity of any body *by experiment:* first support the body, as by a string, at one point, and when at rest extend the vertical line through the body; then suspend it from a second point, and also prolong this vertical line. The point of intersection of these two lines will be the required centre of gravity; for, by the above article, the centre of gravity lies in each of these lines, and will therefore be at their point of intersection. This method is often useful in the case of irregular unsymmetrical bodies, to which the principles already given (163, 167) cannot be applied.

174. If the vertical line through the centre of gravity does not pass through the point or line of support, the body will tend to rotate about this point or axis. For (Figs. 109, 110) the weight of the body ABC, represented by the vertical line Gd, may be resolved into two components, one, Ga, on the line GS drawn to the axis, and the other, Gb, at right angles to it. The first component

that the body tends to return to its original position when displaced slightly. In unstable equilibrium it is at the *highest* possible point, and is lowered by a change of position. In neutral equilibrium it remains at a fixed distance from the support, whatever the position of the body.

176. Stability of a Body resting on a Base. *A body resting upon a base will stand or fall according as the vertical line through the centre of gravity falls within or without the base of support.* By base of support is meant the salient polygon formed by lines joining the extreme points of support. For example, for a table with three legs it is a triangle formed by lines joining their extremities.

This principle is illustrated in Figs. 115, 116, 117,

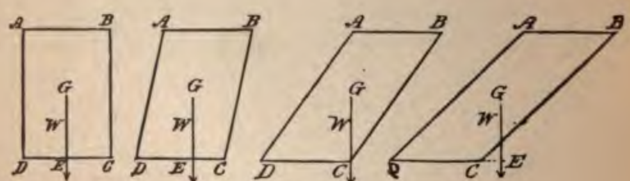

FIG. 115. FIG. 116. FIG. 117. FIG. 118.

118. In each case, let a vertical line through the centre of gravity (G) represent the weight of the body; then, taking the moment (152) of the weight about the point C, which would be the axis in case of an overturn in that direction, the product of $W \times EC$ measures the tendency of the body to *retain* its position (Figs. 115, 116), and the product $W \times CE$, Fig. 118, measures the tendency of the body to *overturn*. In Fig. 117 the line of the weight passes through the axis of rotation; hence its

moment is zero, and the body is on the point of overturning.

In general, when the vertical line of the weight falls within the base, the product of the weight into the perpendicular distance to the nearest side is called the *moment of stability*. When the same line falls without, this product of the weight into the perpendicular distance to the nearest side is called the *moment of instability*.

177. Conditions upon which the Stability of a Body depends. If, as in Figs. 119, 120, a force P acts, as indicated, at the point A, the body will be on the point of overturning when the moments of P and W about C are equal and opposite (156). If, in general, the arm of P is R (here BC), and of W is r (EC), then

$$P \times R = W \times r.$$

Also, if a force P acts to support a body tending to overturn (as, for example, a prop), then the same equation

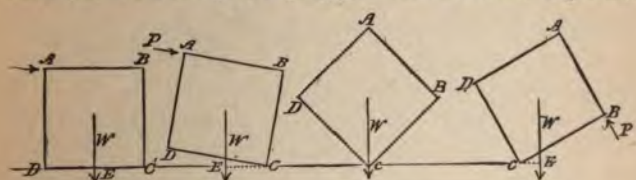

FIG. 119. FIG. 120. FIG. 121. FIG. 122.

will hold true when the body is supported, as shown in Fig. 122, and the value of P given by the equation is the pressure on the prop.

In the first case it is obvious that the greater the overturning force required, the greater the stability of the body. But from the above equation, P must increase as W increases; that is—

[illegible] greater as the weight *increases* [illegible] a stone tower is less [illegible] wooden one of the same dimen-[illegible]

[illegible] increases, that is—

[illegible] support the greater the [illegible] remaining the same. Also, if [illegible] of given area, supposing [illegible] through the centre, the stability [illegible] increases with the number [illegible] the circle.

[illegible] are constant, F is increased as [illegible] in other words—

[illegible] the power the more readily [illegible] of overturning. For exam-[illegible] different heights but otherwise

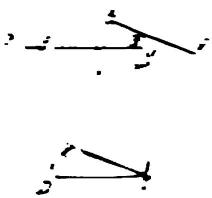

Fig. 22.

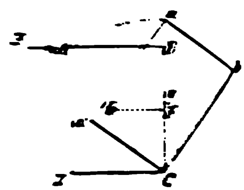

Fig. 23.

[illegible] the more easily blown over, [illegible] the wind would act at a [illegible]

[illegible] increased as the position [illegible] lowered. This relation does [illegible] that applies only to [illegible] where, as it was expressed, the body is on the point of overturning.

In order that the body should actually be overturned P must continue to act (Figs. 123, 124, 125), diminishing continually as r diminishes, and becoming zero when the body is in the second position indicated in each

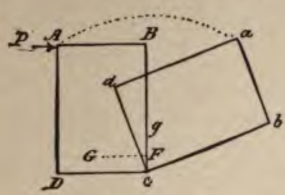

FIG. 125.

figure. Hence work is done in accomplishing this result, and this is estimated most simply by the product of the weight into the distance which the centre of gravity is raised (97). The work done increases as the position of the centre of gravity is lowered. For example, the initial values of P are the same in Figs. 123, 124, and 125, as also the final values ($= 0$), but the arc Aa, through which P acts, and the height Fg, through which the weight is raised, are least for the highest position of the centre of gravity (Fig. 123), and greatest for its lowest position (Fig. 125). Thus, a stage-coach with a heavy load of trunks on top has its centre of gravity high, and is easily overturned by a slight irregularity in the road.

EXAMPLES.

XXV. *Stability.* Articles 172–177.

[The centre of gravity in each case is assumed to be at the geometrical centre.]

1. If the weight of the structure, of which $ABCD$ (Fig. 115, p. 186) is a section, is 150 lbs., also $AB = 6$ feet, $AD = 10$ feet:

What force P will put it on the point of overturning (a) if acting at A? (b) if acting at the centre of AD? (c) If it rests on the side AD and the force acts at B and the middle point of AB respectively, what answers are obtained?

2. In Fig. 122, p. 187, $AD = AB = 10$ feet, the angle $ABC = 90°$ and $BCE = 15°$, the weight is 120 lbs.: What is the pressure on a prop placed (a) so as to act at B at right angles to BC? (b) so as to act at the same point but standing vertically?

3. A rectangular frame $ABCD$ (Fig. 117, p. 186) is racked out of shape. If $AB = 12$ feet, $AD = 18$ feet, what is the angle ADC when it is about to fall?

4. A uniform stone tower, 8 feet in diameter, inclines 1 foot for every 10 feet of vertical height: What is the height of the top when it is about to fall?

5. (a) A rough plane is inclined so that a cube resting on it is about to turn over, it not being able to slide: What is the angle? (b) What is the angle for an isosceles triangle ($AB = AC = 10$, $BC = 16$) if it rests on the side AB?

6. A table 6 feet square stands upon four legs, each of which is 12 inches in from the adjacent edges; its height is 3 feet and its weight 24 lbs.: What is the least force required to put it on the point of overturning if applied at the edge (a) as a horizontal push? (b) as a pressure directly down?

7. A table, having a circular top of 2 feet radius, is supported on three legs placed at the edge and at equal distances from one another; the height is 30 inches and the weight 20 lbs.: What is the least force that will put it on the point of overturning if applied at the top (a) as a horizontal push? (b) as a pressure down? (c) acting vertically upward?

8. What work would be done in overturning a cylindrical column of stone weighing 40,000 lbs., 10 feet high and 4 feet diameter, supposing that the centre of gravity is on the axis (a) at the middle? (b) 1 foot from bottom? (c) 1 foot from top?

CHAPTER VIII.—MACHINES.

178. The MACHINES are mechanical contrivances, by the use of which a force applied at one point is made to act at another with a change in either its direction or intensity, or in both. By means of them, for example, the power may raise a weight much larger than itself, or, on the other hand, it may give to the weight a velocity much greater than its own. In all cases, however, the machine is only an instrument by which mechanical energy is transformed; it never creates energy.

179. The principle of the preceding article was laid down in Art. 98, where it was stated that, as follows from the law of the Conservation of Energy, in every case: "The work done by the power is equal to the work expended upon the weight."

The work done by the force (P) acting is equal to the product of it (or its effective component, $P \cos \beta$*) into the distance (s) through which it acts; that is,

$$P.s, \qquad \text{or} \qquad P \cos \beta . s. \qquad (1)$$

The work done in raising the weight is equal to the pro-

* When the force acts obliquely to the motion of the body, it is immaterial, in the estimation of the work done, whether the product of the *effective component of the force* into the whole distance is taken ($= P \cos \beta . s$), or the product of the whole force into the *effective distance;* that is, the resolved part of the motion in the direction of its own action ($= P . s \cos \beta$).

duct of it (W) into the vertical distance (h) through which it is raised; that is,

$$W.h. \tag{2}$$

If the work is done not against gravity in raising a weight, but against some other force producing a *resistance*, the work done is estimated by the resistance overcome (R) into the effective distance (d); that is,

$$R.d. \tag{3}$$

It is, in general, found convenient to use the term weight as including the resistance, though the true distinction must not be forgotten.

In every machine, according to the principle of work just stated,

$$P.s = W.h, \quad \text{and} \quad P.s = R.d, \tag{4}$$

or

$$\frac{P}{W} = \frac{h}{s}, \quad \text{and} \quad \frac{P}{R} = \frac{d}{s}. \tag{5}$$

The relation (5) may be expressed in this way, that: *The Power is to the Weight (or Resistance) as the distance through which the Weight is raised (or the Resistance is overcome) is to the distance through which the Power acts.*

180. Machines are then employed: (1) Where a small power is desired to raise a large weight or overcome a great resistance. In this case there is said to be a *mechanical advantage;* but as seen from equation (4), if W is greater than P, s, the distance through which P acts, must be as many times greater than h, the distance through which W rises. This is sometimes expressed in this form: *What is gained in power is lost in velocity.*

Also—(2) Where an increased velocity is required; in this case h (or d) will be greater than s, but P must be as many times greater than W. There is then said to be a *mechanical disadvantage*, but, similar to the principle above, *what is lost in power is gained in velocity*.

The cases where the attention is directed, in the use of the machine, solely to the diminished or increased velocity of the motion it transmits, the relation of P to W being overlooked, are obviously included in (1) or (2).

(3) Machines are also occasionally employed where only change in direction is required, and here $P = W$ (or R), and hence $s = h$ (or d).

181. The relation of P to W, established in equation (4), is that which is required in order that the power acting *uniformly* should raise the weight *uniformly*. If the value of the power were greater than that thus required, accelerated motion would ensue; and if less, there would be retarded motion and the system would ultimately come to rest.

The same relation of P and W will hold good if the system is at rest and the power simply supports the weight. The principles of statics make it possible to deduce independently this ratio of $\frac{P}{W}$ on the supposition that the weight is at rest. In the pages which follow, the relation will be deduced by both methods: first in accordance with statics, and second on the principle of work.

182. Virtual Velocities. In the second case given above, the principle of work may be stated in this form: If any machine, in equilibrium under the action of several forces, suffers a slight displacement consistent

with the relations of the parts, then the algebraic sum of the work done by the forces will be zero, and conversely. For such a case as this the velocities are imaginary, and are called *virtual.* This is sometimes spoken of as the principle of VIRTUAL VELOCITIES. This principle is essentially that involved in equation (4) or (5) of Art. 179.

183. The Machines with Friction. In the statements which have been made in regard to the relation of the power and weight in the case of a machine, it has been assumed that the work done by the power was all expended in raising the weight. In practice, however, there are various hurtful resistances to be overcome, chief among which is friction. Hence the work done by the power must always be greater than that which the equation (4) requires. For example, if F represents the force of friction, and l the distance through which it is overcome, then the work done against friction, as shown in Art. 100, is $F.l$, and the equation must then be written

$$P.s = W.h + F.l.$$

The law of the Conservation of Energy still holds good; but as the term $F.l$ increases, the amount of work expended in producing no useful effect, but merely useless heat (110), is increased. Hence, although there is theoretically no limit to the mechanical advantage that may be attained (though always with a proportional loss of velocity) by an appropriately constructed machine or combination of machines, there is practically a limit; for, as the complexity increases, more and more of the power is expended without useful effect.

To the resistance of friction must be added other re-

sistances which are additional drains upon the energy communicated to the machine, and which leave less to be expended in raising the weight. Among these are: adhesion of parts in contact, the stiffness of cords, resistance of the air, want of rigidity in the parts of the machine. In the discussion in the following pages all these resistances are left out of account. The weights of the parts of the machines are also to be neglected unless otherwise stated.

The *modulus* of a machine is the ratio of the amount of work practically done by it to that which theory requires.

184. Simple Machines. The Simple Machines, or Mechanical Powers as they are sometimes called, are as follows: (1) The Lever, (2) Wheel and Axle, (3) Toothed Wheels, (4) Pulley, (5) Inclined Plane, (6) Wedge, (7) Screw.

Of these machines the Wheel and Axle and Toothed Wheels are in fact modifications of the Lever. All of them involve the essential idea of a tendency to rotation about an axis, and hence to deduce the conditions of equilibrium for them the principle of the equality of moments is employed (156). The Pulley is based upon the principle of reduplication, depending on the fact of equal transmission of force by a string; or, in other words, that the tension of a rope at every point is the same (122). The Wedge and Screw are essentially modifications of the Inclined Plane.

I. Lever.

A. General Principle of the Lever.

185. The Lever in its simplest form is a rigid bar capable of being turned about a fixed axis called the *ful-*

crum, and supposed to be acted upon by forces in a plane at right angles to this axis. The bar may have any shape, straight, bent, or curved, and the directions in which the power and weight act may make any angles with it.

186. For all forms of the lever the condition of equilibrium is that stated in Art. 156 for a constrained body only free to move about an axis in a plane at right angles to it. The power tends to produce rotation about the fulcrum in one direction, and the weight in the other. Hence—

If the Power and the Weight are in equilibrium, the moment of the Power must be equal and opposite to the moment of the Weight.

This may also be stated as follows:

The Power is to the Weight as the perpendicular distance from the fulcrum to the direction of the Weight is to the perpendicular distance from the fulcrum to the direction of the Power.

For example, in Figs. 126, 127, 128, where the bar is

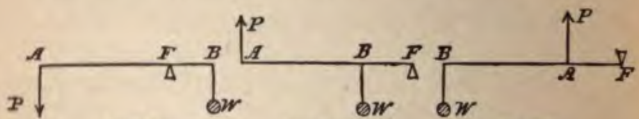

FIG. 126. FIG. 127. FIG. 128.

straight and the power and weight act at right angles to it,

$$P.AF = W.BF,$$

or

$$\frac{P}{W} = \frac{BF}{AF}.$$

The rule holds good equally well when the bar is not straight and the directions are oblique. The positions

of the perpendicular distances from the fulcrum—that is, the arms of P and W—are to be carefully noted in the following figures, 129–134. For all of them the same equation holds good.

The pressure on the fulcrum (the weight of the lever being neglected) in Figs. 126 and 129 is $P + W$, in Fig. 127 it is $W - P$, in Fig. 128 it is $P - W$, and in the other figures it may be calculated by the parallelogram of forces. In the latter cases it is to be noted that, since the power, weight, and resistance of the fulcrum are in equilibrium, their lines of action produced must pass through the same point (158, B).

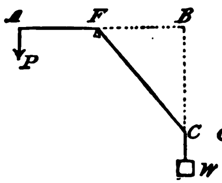

FIG. 129.

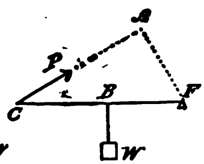

FIG. 130.

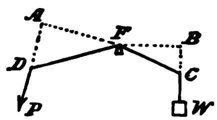

FIG. 131.

In Fig. 132 the lever is curved like an iron pump-handle, the arms of the weight and power are the per-

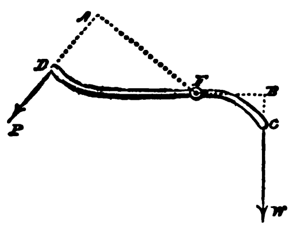

FIG. 132.

pendiculars BF and AF respectively, and the above equation is true:

$$P.AF = W.BF.$$

187. Three Kinds of Lever. The three forms in Figs. 126, 127, 128 are sometimes called the three kinds of lever, though there is no essential difference between them. In the first kind the fulcrum is between the power and weight; if nearer to the latter, there is a mechanical advantage; if nearer to the power, a mechanical disadvantage. If the arms are equal, then $P = W$, as in the ordinary balance (191).

In the second kind the fulcrum is at the end, and the weight nearer to it than is the power; in this case there is always a mechanical advantage.

In the third kind the fulcrum is at the end, but the power is nearer to it than the weight, and there is therefore a mechanical disadvantage.

188. The first form of lever is illustrated by the crow-bar, by means of which, owing to the great difference in the lengths of the arms, a very great resistance can be overcome. Scissors and nippers are double levers of this class, and the handle and claw of a hammer form a curved lever.

The distinction between the gain of power and loss of velocity, and the converse, as determined by the position of the fulcrum, is illustrated by the shears used by a tinman and a tailor respectively. Those of the former have short blades and long handles, and can consequently overcome a great resistance slowly; those of the tailor have short handles and long blades, and move *quickly*, so as to cut yielding materials.

An example of a lever of the second class is a wheel-barrow: the fulcrum is at the centre, or axis, of the wheel, the weight acts down at the centre of gravity of the load and barrow together, and the power is applied at the handles. A nut-cracker or a lemon-squeezer is an example of a double lever of this kind.

The human fore-arm is an example of a lever of the third class: the elbow-joint is the fulcrum, the weight is grasped in the hand, and the power is applied by a tendon from the muscle above attached very near the elbow, and acting obliquely. There is consequently a very serious mechanical disadvantage, but in its place is gained great rapidity of movement. A pair of tongs is another example of a lever of this kind.

189. The following cases involve the principle of the lever. DF (Fig. 133 or 134) is a heavy rod hinged at F, so that it is free to turn in a vertical plane, and supported either by a string carried from C up to E (Fig. 133), or by a prop from C to E below (Fig.

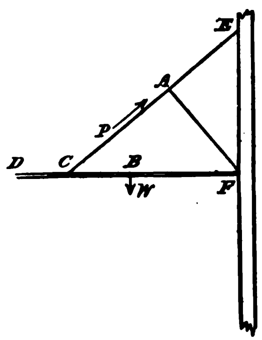

Fig. 133.

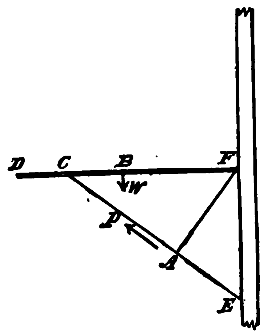

Fig. 134.

134). In this case the power (P) is the tension of the string (or thrust of the prop), and its moment is $P.AF$; the weight is that of the bar acting at its centre of gravity B, and its moment is $W.BF$. The value of P, derived from the equation

$$P.AF = W.BF,$$

will give the tension of the string (or thrust of the prop) needed just to support the rod.

190. The Lever on the Principle of Work. It has been shown in Art. 179 that, in the case of every machine, if friction and all other hurtful resistances are eliminated,

$$P.s = W.h,$$

or
$$\frac{P}{W} = \frac{h}{s}. \tag{1}$$

Here s is the distance through which the power acts, and h the distance through which the weight is raised. If a resistance (R) is overcome through a distance d, then

$$P.s = R.d,$$

or
$$\frac{P}{R} = \frac{d}{s}. \tag{2}$$

By the use of these equations the relation of the Power to the Weight (or Resistance), when the Power raises the Weight uniformly, can be obtained. This value of $\frac{P}{W}$ is the same as that deduced on condition of equilibrium, on the principles of statics.

In the case of the lever, suppose (1) that the power

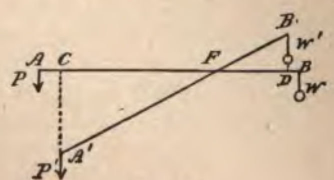

FIG. 135.

always acts vertically (Fig. 135) and turns the lever from the position AB to that of $A'B'$; then the effective

distance through which it acts is $A'C$ ($= s$), and the height through which the weight is raised is $B'D$ ($= h$). Therefore

$$\frac{P}{W} = \frac{B'D}{A'C} = \frac{B'F}{A'F} = \frac{BF}{AF};$$

$$\therefore P.AF = W.BF, \qquad \text{as in Art. 186.}$$

(2) Suppose the power and resistance to act continually at right angles to the straight lever AB (Fig. 136)

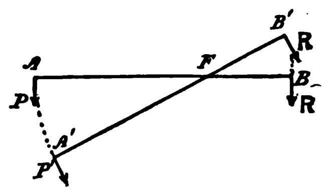

FIG. 136.

while it turns from the position AB to $A'B'$. Here the power acts through the arc AA' ($= s$), and the resistance is overcome through the distance of the arc BB' ($= d$). Hence

$$\frac{P}{R} = \frac{BB'}{AA'} = \frac{BF}{AF};$$

$$\therefore P.AF = R.BF.$$

In general, the relation may be obtained after the same manner, whatever the shape of the lever or the directions of P and W (or R). In each case, however, it must be remembered that s and h (or d) are not necessarily the actual, but always the *effective*, distances.

B. *Some Special Applications of the Principle of the Lever.*

I. BALANCE.

191. The BALANCE is a contrivance used for measuring the mass of bodies; or, in familiar language, of determining their weight by comparison with that of certain assumed units. (See Arts. 54, 55.) In its ordinary form it consists of a beam, so constructed as to be at once strong, rigid, and light. This beam is poised on a knife-edge, in the middle, as a fulcrum, often resting on a plate of agate. From the extremities of the two equal arms are suspended pans of the same size and weight. The object weighed is placed in one pan, and the counterpoise is adjusted to balance it in the other.

192. A good balance must satisfy these three conditions: it must be (1) *true*, (2) *stable*, and (3) *sensible*.

(1) It is *true* when the arms are of exactly the same length and weight, and when the scale-pans are also just equal. It will then be rigidly true that $P = W$. If, however, the arm of the pan in which the object is weighed is longer, then a smaller amount of it will balance the given counterpoise, and the purchaser in such a case would be defrauded, and conversely. This inequality would be proved by exchanging the two objects. If the apparent weight in one pan is a, and in the other b, the true weight will be equal to $\sqrt{ab}$.

(2) The balance must also be *stable*; that is, after being slightly disturbed it must return to its original position. In order to satisfy this condition the centre of gravity must be below the axis on which the beam turns, for if above it would be in unstable equilibrium, and if on the

axis the equilibrium would be neutral; that is, the beam would balance in every position (175).

(3) The balance must be *sensible;* that is, when the weight in one pan slightly exceeds that in the other, this difference must be indicated by the inclined position of the beam when it comes to a state of rest. The sensibility is obviously greater as the angle of deflection increases for a constant difference of weight; this angle is often measured by a long slender rod which is at right angles to the beam and turns with it. The degree of sensibility to be attained in a given case depends upon the object for which the balance is to be used; for example, a balance suitable for use in chemical analyses should indicate distinctly a difference of $\frac{1}{10}$ of a milligram; that is, $\frac{1}{10000}$ of a gram.

193. The conditions upon which the sensibility of a balance depends are: (1) the length of beam, (2) the

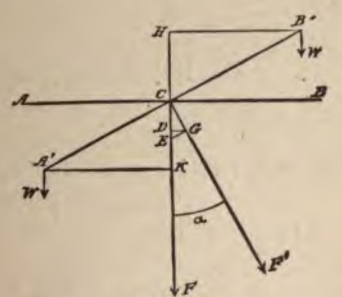

FIG. 137.

weight of the beam, and (3) the position of the centre of gravity.

Let A and B (Fig. 137) be the points of the beam at which the scale-pans are hung, and let C be the axis or

fulcrum; then the line ACB will join these three points. Suppose also that the shape of beam is such that its centre of gravity is at G, at which point its weight (Q) consequently acts. $A'B'$ (Fig. 137) represents the inclined position which the line AB takes for a given difference of weight in the two pans of $W-W'$. The angle of deflection of the rod FCF' ($=\alpha$) is the same as that of the beam. It is obvious that, for a given value of $W-W'$, the greater the angle α the greater the sensibility of the balance. Suppose the whole in equilibrium; then, by taking the moments about C (156),

$$W.A'K = W'.B'H + Q.DG.$$

But since $A'K' = B'H$,

$$(W-W').A'K = Q.DG.$$

From the similar triangles $CA'K$, DGE,

$$\frac{A'K}{A'C} = \frac{DG}{GE}, \quad \text{or} \quad \frac{A'K}{DG} = \frac{A'C}{GE};$$

$$\therefore \quad (W-W')\,AC = Q.GE.$$

But $GE = CG \tan\alpha$; hence

$$(W-W')\,AC = Q.CG \tan\alpha,$$

or

$$\tan\alpha = \frac{(W-W').AC}{Q.CG}.$$

In this final equation AC is one half the length of the beam, and CG is the distance of its centre of gravity below the axis. Now, as has been stated, the sensibility of the balance increases as α increases, for a given value $(W-W')$. It is obvious, from this equation, that $\tan\alpha$ is increased (1) by making AC, the length of the beam,

greater; also, (2) by diminishing Q, the weight of the beam; and finally, (3) by diminishing CG—that is, by bringing the centre of gravity as near as practicable to the axis.

The most satisfactory result will be obtained by considering these conditions together, since they depend upon one another. For example, if the length of the beam is increased, its weight must be also, in order that it still be rigid; again, although the sensibility increases as the distance of the centre of gravity below the axis is diminished, the motion of the beam, as it tends to come to a position of equilibrium, becomes more slow, so that there is also a practical limit in this direction.

The equation shows that the difference in weight is proportional to tan α, and for very small angles it is proportional to the angle itself (α).

II. STEELYARD.

194. Common Steelyard. In the STEELYARD we have, in the place of the fixed arm and varying counterpoise of the ordinary balance, a varying lever-arm and a fixed counterpoise. The bar is made heavier at one extremity, and to this end is attached the hook or scale-pan; near it is the point of support. This axis is consequently near the centre of gravity of the whole, but usually does not coincide with it. In order to graduate the steelyard, it is necessary to determine first the zero-point of the scale, and then the distance to be marked off from it for each unit of weight (*e.g.*, 1 lb.) and fraction of it.

Let AB (Fig. 138) be the steelyard, supported at C. Represent the weight of the whole by Q acting at the centre of gravity G. In order that the bar should balance horizontally about C when there is no weight

on the hook, it is necessary to place the selected counterpoise P at such a point, D, that

$$Q.CG = P.CD.$$

This point D is then the zero of the scale, or the position of P for 0 lbs. at A.

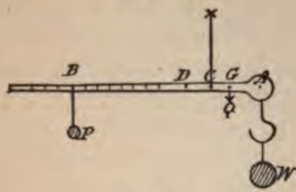

FIG. 138.

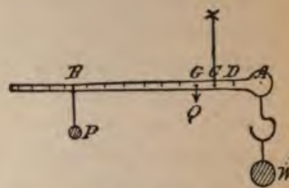

FIG. 139.

Let now a weight W be placed on the hook so that it acts through A; then the counterpoise P will balance it at B, if (156) the moments about C vanish; that is,

$$W.AC + Q.GC = P.CB;$$

or, since $$Q.GC = P.CD,$$

$$W.AC = P.CB - P.CD = P\,(CB - CD),$$

or $$W.AC = P.DB,$$

and $$DB = \frac{W.AC}{P}.$$

If $W = 1$ lb., then the value of DB gives the position of the one-pound notch on the scale, and at twice this distance from D will be the two-pound notch, and so on.

If, as in Fig. 139, the centre of gravity is on the other side of the fulcrum, the position of the zero-point D will also be changed, but the value of DB is obtained in essentially the same way.

195. A form of the steelyard as actually employed is seen in Fig. 140. It will be observed that both sides of the bar are graduated, and moreover there is a second

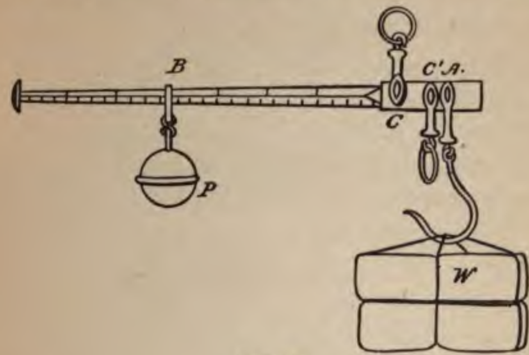

Fig. 140

ring to support it at C'. When the steelyard is turned over and supported at C', the weight has a shorter lever-arm, and consequently, the counterpoise remaining the same, with this second graduation the instrument is adapted for heavier weights than in the first case.

Fig. 141.

A common form of the steelyard is also seen in the post-office scales, where the object to be weighed is placed on a platform, and the counterpoise slides along the graduated arm.

Another very simple form of balance involving the same idea of a varying lever-arm is seen in the contrivance often employed for weighing letters (Fig. 141). When there is no weight at B, the weight of the instrument acts through its centre of

gravity directly below the point of suspension C. If now a letter is placed between the springs at B the position is slightly changed, so that the moment of its weight is equal to the moment of the weight of the instrument in its new position.

196. Danish Steelyard. In the Danish steelyard no counterpoise is employed, but the adjustment is made by shifting the position of the supporting-hook, and consequently giving the weight of the bar a longer or shorter lever-arm. Let (Fig. 142) AB represent the

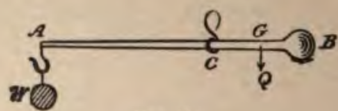

FIG. 142.

bar, heavier at the end B, and let its weight Q act at G. Suppose a weight W to be hung on the hook at A; then, in case of equilibrium, we have

$$Q.CG = W.AC;$$

but $$CG = AG - AC,$$

$$\therefore\ Q(AG - AC) = W.AC, \text{ or } (Q + W)\,AC = Q.AG;$$

$$\therefore\ AC = \frac{Q.AG}{Q + W}.$$

The arm is graduated by letting $W = 1$ lb., 2 lbs., etc., in succession; since Q and AG have constant values, AC is thus obtained for each case.

197. Roberval's Balance. In many forms of balance in common use, instead of two scale-pans suspended

from a beam above, there are two platforms supported from beneath, upon one of which is placed the object to be weighed, and upon the other the counterpoise; or (as in the post-office scales alluded to in Art. 195) there is one platform, and the place of the other is taken by a graduated arm upon which slides a constant counterpoise. In such balances it is essential that the indications should be accurate, no matter what the position of the load on the platform.

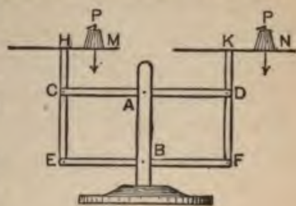

Fig. 142*a*.

The way in which this end is often attained is illustrated by *Roberval's balance* (Fig. 142*a*); CD, EF are here two bars of equal length, pivoted to the upright support at A and B; they are also jointed at C, E, and D, F, thus forming a rectangular frame. The equal pans are supported at H and K, and upon them respectively are placed the object to be weighed and the counterpose, as P, P at M and N. (In actual use the frame $CDFE$ is generally concealed in the stand of the balance.)

In the figure it is seen that the weights P, P are at very unequal distances from the axis AB, but the accuracy is not impaired by this fact. To prove this, suppose two opposite forces, each equal to P, to act at K, and two others similar at H; they will not alter the previous conditions. We have now, in place of P at N, a force equal to P acting downward at K, and a couple (150) whose moment is $P.KN$; also, in place of P at M, we have an equal force acting downward at H, and a couple whose moment is $P.HM$. The two equal downward forces at H and K will obviously balance each

other; the couples though unequal do not disturb the equilibrium, for they merely produce unequal strains at the fixed points A and B, and do not alter the effect of the other forces.

III. TOGGLE-JOINT.

198. Toggle-Joint. Fig. 143 represents two combined levers, AB, BC, forming what is called a TOGGLE-JOINT. They are hinged together at B, forming an angle $ABC = 2\alpha$; further, the lever AB turns freely at A, while the end C of BC is free to move in the direction xy, and acts against the resistance Q. Suppose the

FIG. 143.

force P to act vertically downward at B against the resistances R and R; if the system is in equilibrium, we have, by Art. 133,

$$\frac{P}{R} = \frac{\sin ABC}{\sin PBC} = \frac{\sin 2\alpha}{\sin \alpha} = \frac{2 \sin \alpha \cos \alpha}{\sin \alpha} = 2 \cos \alpha.$$

But the effective resistance Q is only one component of R, the other being supplied by the reaction of the plane xy; hence

$$Q = R \sin \alpha, \qquad \text{and} \qquad R = \frac{Q}{\sin \alpha};$$

$$\therefore \quad \frac{P}{Q} = \frac{2 \cos \alpha}{\sin \alpha} = \frac{2}{\tan \alpha}.$$

As now the levers AB, BC straighten out, the angle 2α approaches $180°$, and tan α approaches infinity as its limit; the mechanical advantage, therefore, when the levers are nearly in a straight line is enormously great, and hence a very great resistance can be overcome. The toggle-joint is seen in the arrangement by which the cover of a carriage is raised.

199. The principle of the toggle-joint and its great efficiency are well illustrated by the Stone-Crusher invented by Mr. Blake, of New Haven. The accompanying cut (Fig. 144) gives a longi-

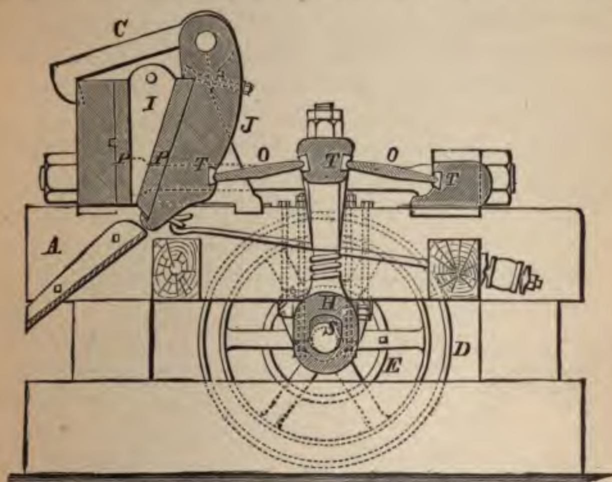

FIG. 144.

tudinal section of the machine. The parts show so clearly the relations explained in the preceding article that but little description is needed. The power is applied to the wheel E, and as it revolves the central post alternately rises and falls. This serves to work the levers, or toggles, O, O. One end of them is stationary on the right (corresponding to A in Fig. 143), and the other acts on the movable iron mass J swung on K. Finally this re-

sults in the opening and shutting of the jaws P, P. The distance through which the movable jaw works is small, and the force exerted against any object placed between is enormous. In use the blocks of stone or ore are fed in from above, and the motion of the jaws rapidly crushes them down to uniform fragments of a size regulated by the distance between the jaws at the lowest points; the fragments pass out by the shute A. Such a machine will yield about 10 tons of broken rock per hour.

IV. COMPOUND LEVERS.

200. It is seen in Art. 186 that in the lever, by making the arm of the weight very short and that of the power very long, any required mechanical advantage may theoretically be obtained. In practice, however, various difficulties would obviously arise in an attempt to gain power in this way. To avoid them, and at the same time to have greater compactness, it is found better to employ a series of levers, in which the weight of the first becomes the power of the second, and so on.

In Fig. 145, let AC, DF, GK be three levers, arranged as just indicated. Q is at once the weight of the

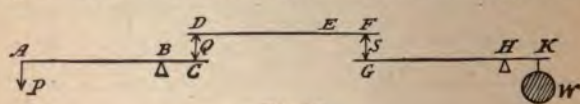

FIG. 145.

first and the power of the second lever, and the same is true for S with reference to the second and third levers.

Now if the whole is in equilibrium:

$$\frac{P}{Q}=\frac{BC}{BA}, \qquad \frac{Q}{S}=\frac{EF}{ED}, \qquad \frac{S}{W}=\frac{HK}{HG}.$$

By multiplying these ratios together,

$$\frac{P}{W}=\frac{BC}{BA}\times\frac{EF}{ED}\times\frac{HK}{HG}.$$

It is seen here that

The final mechanical advantage is equal to the product of the several values for each successive lever. This principle is true not only for levers in combination, but in general for any compound machine.

In determining the ratio of $\frac{P}{W}$ for a compound machine on the principle of work, it is to be noted that the essential point is to know the distances through which P and W act. If these can be determined, their inverse ratio gives the ratio required, and the intermediate steps in the machine are of no importance.

201. The principle of the compound levers finds an important application in the scales used for weighing very heavy objects, as, for example, hay-scales, or railroad scales for cars loaded with coal (say 10 tons each). By a combination of a series of levers, and at the same time by the suitable distribution of the weight brought on the platform, this being supported at a number of points, very heavy weights may be determined with all desirable accuracy. The whole is balanced by a counterpoise, or series of them, used in connection with a steelyard arm.

EXAMPLES.

XXVI. *Lever.* Articles 185–190.

[The weight of the lever is to be neglected, except when otherwise stated.]

1. The force $P = 40$ lbs. acts as in Fig. 126, p. 196; $AF = 8$ feet and $AB = 10$ feet: What weight can be supported ?

2. If (Fig. 127, p. 196) $AB = 10$, $BF = 2$, and the weight is 120 lbs., what force P is required to support it ?

3. If (Fig. 128, p. 196) $AB = 14$, $AF = 2$, and $P = 100$ lbs., what weight can P support?

4. What is the pressure on the fulcrum in each of the above cases?

5. AFC is a bent lever (Fig. 129, p. 197); $AF = 14$, $FC = 16$, $AFC = 135°$, $P = 30$ lbs.: What is W?

6. CFD is a bent lever (Fig. 131, p. 197); $CF = 18$, $FD = 12$, $PDF = 150°$, $FCW = 165°$, and $W = 60$ lbs.: What is P?

7. If in Fig. 130, p. 197, $CB = 16$, $BF = 2$, and $ACF = 30°$, also $P = 40$ lbs., what is W?

8. What is the pressure on the fulcrum in examples 5 and 7?

9. A heavy uniform rod DF (Fig. 133, p. 199), weighing 25 lbs and 2 feet long, is hinged at F; it is supported by a string carried from C ($CF = 20$ in.) to a point E, 12 inches vertically above F: What is the tension of the string?

10. A heavy uniform shelf DF (Fig. 134), 18 in. wide ($= DF$), weighing 36 lbs., and hinged at F, is supported by a prop carried from C ($CF = 12$ in.) to a point E below F, so that $CF = FE$: What pressure does this prop feel?

11. A uniform stick 8 feet long, weighing 2 lbs., is supported between the thumb and first finger; the one acts at the extremity as a fulcrum, and the other as a force at right angles an inch from it: What is the force required when the stick is horizontal? when inclined 60° to the horizontal?

12. A rod weighing 10 lbs. has a weight of 10 lbs. at one end and of 20 lbs. at the other: Where must the fulcrum be in case of equilibrium?

13. Forces of 8 and 12 lbs. act at the extremities of a bar 16 feet long, and in directions making angles of 135° and 150° respectively with it: Where is the fulcrum in case of equilibrium?

XXVII. *Balance.* Articles 191–193.

1. A body is equivalent to a weight of 12 lbs. in one pan of a false balance, and of $16\frac{1}{3}$ lbs. in the other: What is the true weight?

2. A body is equivalent to a weight of 6 lbs. 4 oz. from one arm of a false balance, and of 4 lbs. 6 oz. from the other: What is the ratio of the lengths of the arms?

3. The true weight of a body is 15 oz., its apparent weight in one pan of a balance is 1 lb.: What would it seem to weigh in the other pan?

XXVIII. *Steelyard.* Articles 194–196.

[The common steelyard (194) is intended unless otherwise stated.]

1. The longer arm of a steelyard is 26 inches in length, the shorter $2\frac{3}{8}$ inches; the arrangement of the scale-pan (or hook) is such that $P\,(=2$ lbs.$)$ at B (Fig. 139, p. 206), if $BC = 10$ inches, balances 8 lbs. at A: (*a*) Where is the zero of the scale? (*b*) If the whole weighs $1\frac{1}{2}$ lbs. $(= Q)$, where is the centre of gravity? (*c*) What must be the graduation for ounces? (*d*) If P cannot be conveniently brought nearer than $\frac{3}{8}$ of an inch to C, what are the greatest and least weights for which it can be used?

2. The whole length of a steelyard is 24 inches; CG (Fig. 138) $= \frac{1}{4}$ in., $CA = \frac{3}{4}$ in., $P = 8$ oz., and $Q = 1$ lb.: (*a*) Where is the zero of the scale? (*b*) What is the length of graduation for 1 lb.? (*c*) How large weights can it be used for?

3. In Fig. 140, p. 207, $AC = 3$ in. and $AC' = 1$ in.; the counterpoise $= 12$ oz.: What is the length of a division on the scale for 1 oz. in each position?

4. The weight of the beam of a steelyard is 3 lbs., and the distance of its centre of gravity is $\frac{1}{4}$ inch from the fulcrum: Where must a counterpoise of 1 lb. 12 oz. be placed to balance it?

5. The length of a Danish steelyard is 30 in., its weight is 4 lbs., and acts at a point 3 in. from one end; a body weighing 12 lbs. hangs at the other end: Where is the fulcrum?

6. The length of a Danish steelyard is 28 in., its weight is 3 lbs., acting at a point 4 in. from one end: (*a*) Where is the 1-lb notch? (*b*) the 2-lb.?

II. Wheel and Axle.

202. The Wheel and Axle, in its simplest form, consists of two cylinders of different sizes, rigidly connected and turning about a common axis; the larger is called the wheel, and the smaller the axle. The power is applied to the end of the rope wound about the wheel, and the weight is raised by a rope wound upon the axle. This is seen in Fig. 146.

A good example of this principle is seen in the *fusee* (Fig. 150), which is applied to some watches and clocks.

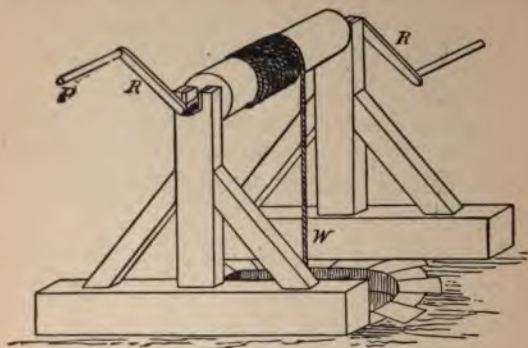

FIG. 148.

As the spring unwinds its force diminishes; but by means of the fusee it is made to act on a continually increasing lever-arm, and by a proper adjustment the moment, or turning power of the force, can be kept constant.

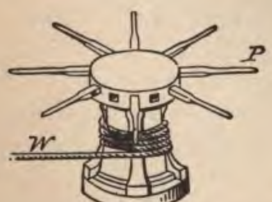

FIG. 149.

FIG. 150.

The wheel of a vehicle is useful in reducing frictional resistance, as explained in Art. 86; in overcoming obstacles in the road it acts as a continuous lever, hence the advantage of a large wheel.

208. Chinese Windlass. The combination of the wheel and axle seen in Fig. 151 is sometimes called the *Chinese Windlass*, or the differential windlass. There are here two axles of different sizes, and the arrangement is such that as the rope is wound up on the larger axle it is unwound on the smaller one, it passing under a movable pulley which supports the weight. The upward ascent of the weight is consequently very slow, but the mechanical advantage very great. Let R be the radius of the crank-arm,

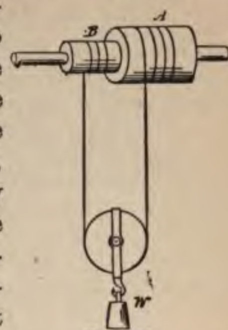

FIG. 151.

and a and b respectively those of the larger and smaller axles. The tension of the rope is obviously $\frac{1}{2}W$. If the machine is in equilibrium, then, since the rope tends to turn the smaller axle in the same direction, and the larger in the opposite direction, to the power, by the equality of moments:

$$P.R + \tfrac{1}{2}W.b = \tfrac{1}{2}W.a,$$

$$P.R = \tfrac{1}{2}W\,(a - b),$$

$$\frac{P}{W} = \frac{a-b}{2R}.$$

This result, also easily obtained by the principle of work (206), shows that the mechanical advantage becomes very great as the difference between the size of the axles is diminished.

EXAMPLES.

XXIX. *Wheel and Axle.* Articles 202–208.

1. The radius of the axle is 2 inches, that of the wheel is $2\frac{1}{2}$ feet, and the power acting is 80 lbs.: What weight is supported?

2. A horse exerting a force of 800 lbs. walks in a circle having a diameter of 18 feet and turns, by means of a lever-arm, a vertical post about which a rope is wound: If the diameter of the post is 8 inches, what resistance (*e.g.*, that of a building which is being moved) can the horse overcome?

3. Four men, each exerting a force of 60 lbs. acting on separate lever-arms, 4 feet long, turn a capstan; the radius of the circle about which the rope is wound is 6 inches: What is the pull felt upon the anchor?

4. A weight of 500 lbs. hangs by a rope 1 inch in thickness; $r = 8$ in., and $R = 4$ feet; the power acts on a lever-arm without a rope: What is P?

5. A power of 12 lbs. balances a weight of 200 lbs.; the radius of the axle is 3 inches: What is the diameter of the wheel?

6. In Fig. 148, $R = 18$ in., and the weight of 250 lbs. rises 2 feet while the power makes 5 revolutions: What is P?

III. Toothed Wheels.

209. A Toothed Wheel is a circular disc provided with teeth on the circumference; such a wheel turning

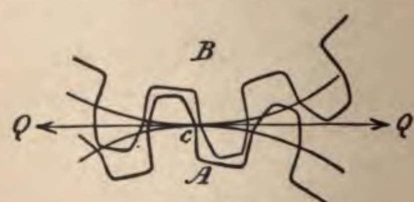

Fig. 152.

on one axis interlocks with a second turning on another axis (Fig. 152), and in this way the force applied at

the first is communicated to the second. There may be a mechanical advantage with a corresponding loss of speed, or a gain in velocity and a consequent mechanical disadvantage.

When the wheels are small, the teeth are nearly rectangular in form. When the wheels are very large and great force is employed, the shape of the teeth is a matter of essential importance, in order that the loss of power arising from their mutual friction and resistance

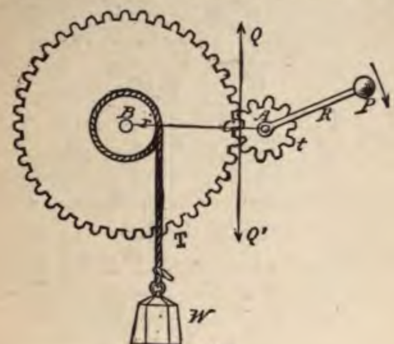

Fig. 153.

may be a minimum. The detailed development of this subject belongs to applied mechanics.

As seen in Fig. 152, when the wheels are turned, the teeth roll upon one another so that the points in which their mutual resistance is felt, due to the power and weight acting, lie very nearly at the touching points of the two circles drawn. These ideal circles are called the pitch-circles, and are concentric with the wheels themselves respectively.

210. Relation of P to W. In Fig. 153, let the power (P) act on the radius R; while the weight (W), supported

by the rope, acts on the radius r, they tend to turn their respective wheels in the direction of the arrows. The resistance between the two wheels is felt (209) in the line QQ', and if the system is in equilibrium, by the principle of the lever (156), the moment of Q' about A is equal and opposite to that of P, and of Q about B to that of W. Hence the relations:

$$P.R = Q'.AC,$$

and
$$W.r = Q.BC;$$

that is,
$$\frac{P.R}{W.r} = \frac{Q'.AC}{Q.BC};$$

$$\therefore \frac{P.R}{W.r} = \frac{AC}{BC} = \frac{2\pi AC}{2\pi BC}.$$

But since the number of teeth in the two wheels is proportional to their circumferences, if $t =$ the number of teeth in the power-wheel, and T those in the weight-wheel, we have

$$\frac{P.R}{W.r} = \frac{t}{T}.$$

This may be stated:

The moment of the Power is to the moment of the Weight as the number of teeth in the Power-wheel is to the number of teeth in the Weight-wheel.

The final equation above may be written:

$$\frac{P}{W} = \frac{r}{R} \times \frac{t}{T}.$$

This is another application of the principle explained in Art. 200, since, in the case supposed, the machine is really compounded of the wheel and axle and the toothed wheels.

211. Toothed Wheels on the Principle of Work. Suppose (Fig. 153) that the power continues to act through one circumference of its lever-arm, $2\pi R$ $(= s)$; if no toothed wheels intervened, the weight would rise through a distance equal to the circumference of the axle, $2\pi r$ $(= h)$. But the ratio of the distances through which the toothed wheels will turn is the inverse ratio of their number of teeth; that is, if the smaller wheel has 20 and the larger 40 teeth, the former will revolve twice $\left(\frac{40}{20}\right)$, while the other turns once; or, in general, $\frac{T}{t}$. Therefore

$$\frac{P}{W} = \frac{2\pi r}{2\pi R} \times \frac{t}{T} = \frac{r}{R} \times \frac{t}{T}, \quad \text{as above.}$$

212. Application of Toothed Wheels. An illustration of the use of this machine is seen in Fig. 154, to which

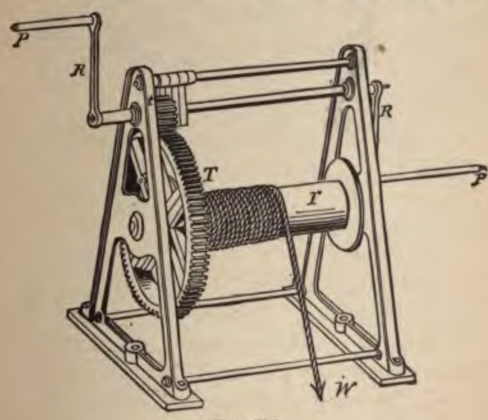

FIG. 154.

the final equation of Art. 210 applies. Severa. series or "trains" of toothed wheels are employed in derricks

and cranes for raising very large weights. One of these, with three pairs of toothed wheels, is represented in Fig. 155. The relation of the power to the weight here, by the preceding principles, is:

$$\frac{P}{W} = \frac{r}{R} \times \frac{t}{T} \times \frac{t'}{T'} \times \frac{t''}{T''}$$

For example, if $P = 10$ lbs., the radius of the axle $(r) = 3$ inches, that of the crank-arm $(R) = 2\frac{1}{2}$ feet; if, also, the number of teeth in each of the smaller wheels (t, t', t'') is 20, and of the larger wheels (T, T', T'') 120, then the weight which could be raised, all hurtful resistances being left out of account, would be 21,600 lbs., or about 10 tons.

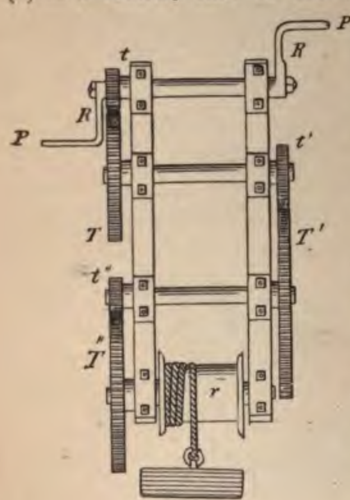

Fig. 155.

213. An excellent illustration of the use of toothed wheels, where a "gain in power" is required, is afforded by the "back gears" of a large turning-lathe. The arrangement is such that the cone-pulley F (Fig. 156), by which the power is applied, turns independently of the wheel L and of the spindle of the lathe, which last are attached together. The "back gears" are two toothed wheels attached to a common axis, and so placed behind the wheels G and L, respectively, that by a slight adjustment they can be, when required, put in gearing with G and L. When the simple motion of the lathe only is needed, a pin connects the cone-pulley F with L, and then the spindle is turned directly at a

rate depending, as mentioned in Art. 216, on the pulleys over which the belt passes. If, however, a greater resistance is to be overcome, as when a large object is to be turned, the pin connecting F and L is taken out, and the "back gears" connected with G and L. The motion of the wheel G is then communicated to the larger wheel behind; this gives to the axis of the latter a speed reduced in the ratio of the number of teeth of the two. This axis turns the second small wheel behind L, and this gives motion, reduced in rate as before, to L and the spindle connected with it. If the number of teeth in the two pairs of wheels are respectively 20 and 80, then the speed of the spindle is reduced by this contrivance 16 times, and a corresponding mechanical advantage is gained.

214. The *rack and pinion* consists of a straight bar in which teeth are cut, into which fits the toothed wheel, which is revolved by a handle or screw-head. By this means the bar, and that to which it is attached, is raised or depressed. This arrangement is often employed in practice; for example, in moving the tube of a microscope up or down.

215. Toothed wheels are extensively used in the works of a clock. The point practically considered is here the relative velocity to be given to the successive axes; the relation of P to W is not taken into account.

216. Use of Belts. In machines the motion of one axis is communicated to another by the use of belts, as well as by toothed wheels; there may or may not be a change of velocity. The use of the belt or strap depends on the friction of the surfaces in contact (85).

The velocities of the two axes, assuming that the strap does not slip at all, are in the inverse ratio of the radii of the wheels, and the mechanical advantage is in the direct ratio, as was true of the toothed wheels.

Thus, in the case of the cone-pulley (E, Fig. 156) on

the shaft communicating the power in the shop, and that (F) of the lathe below, the velocity of the axis of the lathe will be greatest, and the power of overcoming

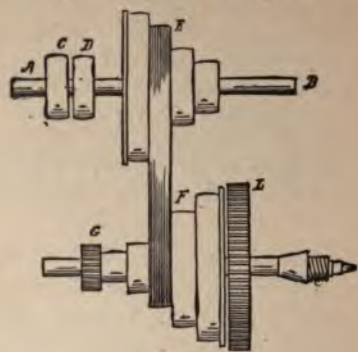

FIG. 156.

resistance the least, when the belt passes over the largest wheel of the latter, and conversely.

In general, according as the straps are or are not crossed, the motion of the second wheel is in the opposite or the same direction as that of the first.

IV. PULLEY.

217. The PULLEY consists of a circular wheel turning about an axis which is attached to a surrounding frame, called the block. About the circumference of the wheel, which is grooved, passes a rope, and at one end of this the power acts. Sometimes two wheels are placed side by side, as in Fig. 157.

The necessity of making the wheel turn on its axis arises from the friction, which is very much diminished in this way; except for this, fixed pegs would answer as well.

The efficiency of the pulley is based upon the principle (122) that *the tension of a given string is the same at every point.*

218. Single Fixed Pulley. *In the single fixed pulley*

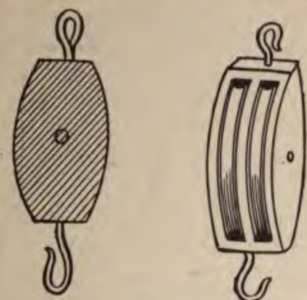

FIG. 157.

the Power is equal to the Weight. This relation follows immediately from the principle stated above, for (Fig. 158) the tension of the rope on both sides of the wheel A must be the same, and to satisfy this condition we must have

$$P = W.$$

There is therefore no mechanical advantage in the use of the single fixed pulley, but it serves to change the direction of the force applied. The tension on the beam at a is equal to $2P$.

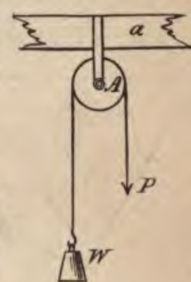

FIG. 158.

219. Single Movable Pulley with Parallel Strings. *In the single movable pulley with parallel strings the Weight is twice the Power.* The tension on both sides of the wheel A (Fig. 159) must, as before, be equal to P;

therefore W is supported by two upward forces, each equal to P, and

$$W = 2P, \qquad \text{or} \qquad P = \tfrac{1}{2}W.$$

If, as in Fig. 160, the rope passes over a second fixed

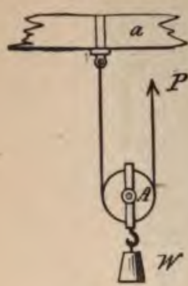

Fig. 159.

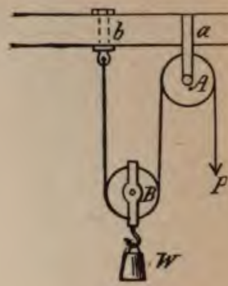

Fig. 160.

pulley, no change is made by this (218), for it is still true that $W = 2P$. The tension at b is P, and at a (Fig. 160) it is $2P$.

220. Single Movable Pulley with Inclined Strings. It was assumed in the preceding article that both branches of the rope were parallel; if, however, they are inclined at an angle 2α (Fig. 161), then

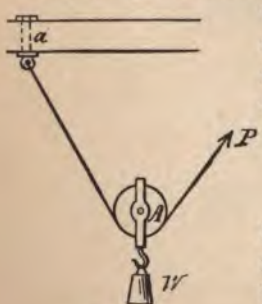

Fig. 161.

$$W = 2P \cos \alpha.$$

In Fig. 161 the tension of the rope on both sides of the pulley is, as before, P, but here W is supported not by two forces each equal to P, but only by their components acting vertically upward. Since (131, a) the vertical line bisects

the angle 2α, these components are equal, and each has the value $P\cos\alpha$; hence

$$W = 2P\cos\alpha.$$

There is evidently a mechanical disadvantage here as compared with the preceding case, for $W = 2P$ only when $\alpha = 0°$, the two strings being parallel, and P increases as α increases. When $\alpha = 60°$ $(2\alpha = 120°)$, then $P = W$; and when $\alpha = 90°$ $(2\alpha° = 180°)$ and the rope is horizontal, $P = \infty$; that is, if the rope were perfectly flexible, no finite force could draw it out horizontal.

FIG. 162.

221. Combinations of Pulleys. *First System.* Fig. 162 represents what is called the first system of pulleys; here $W = 2^n P$. The pulley A is supported by two forces, each equal to P; these act on the string which passes under the pulley B, so that its tension is $2P$. The pulley B is consequently supported by two forces, each equal to $2P$. Again, the tension of the string passing from B under C is 2^2P; and C is supported by two equal forces, each equal to 2^2P; similarly for D; and finally, the pulley E is acted upon by two upward forces, each equal to 2^4P. Hence

$$W = 2^5P,$$

or, in general, $$W = 2^nP,$$

where n is the number of the pulleys.

The beam supports at a a tension of P, also $2P$ at b, $4P$ at c, $8P$ at d, and $16P$ at e.

222. *Second System.* $W = nP$. Fig. 163 represents another system of pulleys. As here but one string is involved, its tension throughout—that is, each of the six branches—is equal to P. The weight is therefore supported by six forces, each equal to P; that is,

Fig. 163.

$$W = 6P,$$

or, in general,

$$W = nP,$$

where n is the number of strings rising from the movable pulleys.

This form of pulleys is the one which is generally employed, though in practice, as remarked in Art. 226, the wheels are placed side by side in two blocks.

If in this arrangement the weight w of the movable block is taken into account, the relation is then

$$W + w = nP.$$

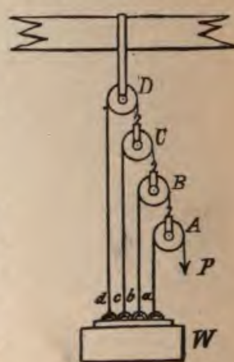

Fig. 164.

223. *Third System.* In the system of pulleys shown in Fig. 164, $W = (2^n - 1)P$.

The tension of the string on which the power acts, that is at a, is P; hence the pull on the wheel B is $2P$, and this force is felt upward at b; still again, the pull on C is $4P$, or 2^2P, and

the same upward at c; also, on D it is $8P$, and the same upward at d. That is, the weight is supported by the four forces; viz., $P + 2P + 2^2P + 2^3P = 15P$. Hence

$$W = 15P = (2^4 - 1)P,$$

or, in general,

$$W = (2^n - 1)P.$$

The tension on the beam is equal to $W + P$, or 2^4P.

224. The following are other forms of pulleys, for which the relations of P to W can be established in the same manner as for those already explained.

In Fig. 165, $W = 4P$. In Fig. 166, $W = 5P \cos \alpha$ (the tendency of the horizontal component of P is also to be noted). In Fig. 167, $W = 81P = 3^nP$.

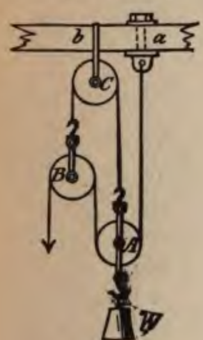

Fig. 165.

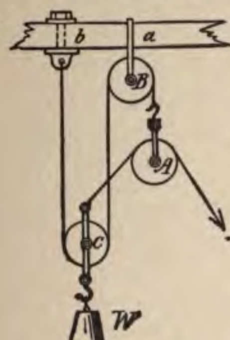

Fig. 166.

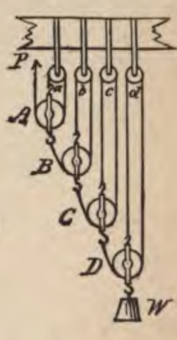

Fig. 167.

225. The Pulley on the Principle of Work. 1. *Single Fixed Pulley.* Here (Fig. 168) the power and weight act through equal distances; that is, $s = h$, and therefore $W = P$.

2. *Single Movable Pulley with Parallel Strings.* In

this case (Figs. 169, 170) the distance through which P acts is twice the height through which the weight rises,

$$\therefore s = 2h, \quad \text{and} \quad W = 2P, \quad \text{or} \quad P = \tfrac{1}{2}W.$$

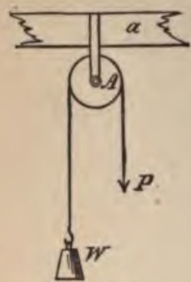

Fig. 168.

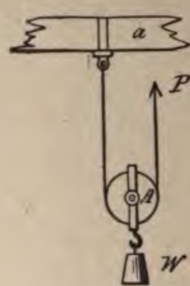

Fig. 169.

Fig. 170.

3. *Single Movable Pulley with Strings not Parallel.* The effective component of the power is $P \cos \alpha$ (Fig. 171), and since $s = 2h$, as above,

$$W = 2P \cos \alpha.$$

4. *First System of Pulleys.* Suppose that the weight (Fig. 172) is raised through a height h, the wheel D is evidently raised $2h$; the wheel C, 2^2h; the wheel B, 2^3h; the wheel A, 2^4h; consequently the power P must act through a distance $s = 2^5h$; that is, $W.h = 2^5h.P$, and $W = 2^5.P$; or, in general,

$$W = 2^n.P.$$

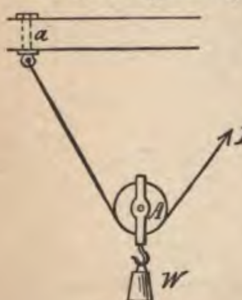

Fig. 171.

5. *Second System of Pulleys.* If the weight is raised (Fig. 173) a height h, each of the 6 strings must be shortened by an equal amount; consequently the distance s through which the power

must act is equal to $6h$. Hence $P.6h = W.h$, and $W = 6P$; or, in general,

$$W = nP.$$

6. *Third System of Pulleys.* Suppose that the weight is raised (Fig. 174) a height h; then, since the pulley D is fixed, the pulley C will on this account move down a distance h; but, since the point c also rises a height h,

FIG. 172.

FIG. 173.

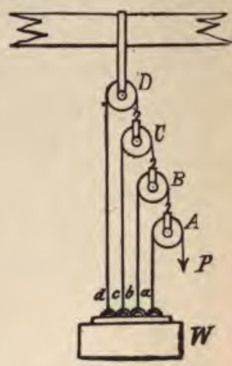

FIG. 174.

the pulley B will move through a distance $2(h) + h$. Similarly, the motion of A will be through a distance $2(2h + h) + h$, and the distance $(= s)$ for $P = 2(2^2h + 2h + h) + h = 15$, or $(2^4 - 1)h$. Therefore $Wh = (2^4 - 1)h.P$, and $W = (2^4 - 1)P$; or, in general,

$$W = (2^n - 1)P.$$

226. Application of the Pulley. The second system of pulleys (222) is that which is most frequently em-

ployed in practice. For the sake of compactness the wheels are arranged side by side in each of the two blocks, as shown in Fig. 175. The relation of P to W is the same as in Fig 163.

FIG. 175.

The theoretical relation of P to W is not practically attained, for both friction and the stiffness of the rope are very serious resistances to be overcome. For many purposes, notwithstanding this loss, the pulley is a most useful mechanical contrivance. It is often used in connection with the wheel and axle or with toothed wheels, as in derricks and cranes. It plays an important part in the rigging of a ship. The single fixed pulley (218) is often employed where it is desired to change the direction of the force; *e.g.*, in the case of a well.

EXAMPLES.

XXX. *Pulley.* Articles 217–226.

1. A man weighing 150 lbs. sits on a platform suspended from a movable pulley (B, Fig. 160, p. 228), and raises himself by a rope passing over a fixed pulley (A): Supposing the strings all parallel, (*a*) what force does he exert? (*b*) What upward force is needed if the rope passes under a pulley fixed to the ground before coming to his hand?

2. In a combination of pulleys, as in Fig. 162, $W =$ 1152 lbs. and $P = 72$ lbs.: How many pulleys are there?

3. In a combination of pulleys, as in Fig. 163, $W =$ 336 lbs. and $P = 42$ lbs.: How many movable pulleys are there?

4. In a combination as in Fig. 164, $W = 840$ lbs., $P = 56$ lbs.: What is the number of pulleys?

5. In the single movable pulley, $P = 100$ lbs.: Calculate the value of W if $2\alpha = 30^\circ, = 60^\circ, = 120^\circ, = 150^\circ, = 180^\circ$.

6. What force is needed to support 500 lbs. by the first system of pulleys, there being 4 in all? What is the force if each pulley weighs $\frac{1}{2}$ lb.?

7. Find P as in example 6, if the second system of pulleys is employed.

8. Find P as in example 6, if the third system is used.

9. What is the relation of P to W in Fig. 165, if the weights of the movable pulleys are taken into account?

10. What is the relation of P to W in Fig. 167, if the weights of the movable pulleys are considered?

V. Inclined Plane.

227. The Inclined Plane, considered as one of the simple machines, is a rigid plane inclined to the horizon at an angle α, and upon it a weight is supported by a power acting in some definite direction. If a section be made perpendicular to the plane, the figure below (176) is obtained. Here HL is the length (l) of the plane,

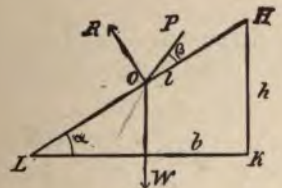

Fig. 176.

HK its height (h), and LK its base (b). The three forces acting upon a body, and, as we suppose, holding it in equilibrium, are the weight (W) acting vertically downward, the power (P) acting at some angle β with the plane, and the resistance of the plane acting at right angles to it; these forces are supposed to act in the same plane.

228. Relation of P, W, and R. *First Method.* If the three forces P, W, R, acting together at O, are in equilibrium, then (133) each force is proportional to the sine of the angle between the directions of the other two. That is,

$$P : W : R = \sin WOR : \sin POR : \sin WOP,$$

$$= \sin (180^\circ - \alpha) : \sin (90^\circ - \beta) : \sin (90^\circ + \alpha + \beta),$$

$$= \sin \alpha : \cos \beta : \cos (\alpha + \beta).$$

Hence $$P = \frac{W \sin\alpha}{\cos \beta}, \quad R = \frac{W \cos (\alpha + \beta)}{\cos \beta}.$$

Second Method. The above relation may also be obtained as follows: Since the forces P, W, R are in equilibrium (141), the algebraic sum of their components along any two lines at right angles to each other will be equal to zero.

Take as these directions (Fig. 177) a line parallel to the length of the plane, and one perpendicular to it

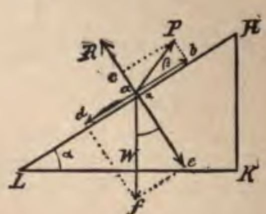

Fig. 177.

coinciding with the direction of R. Then, taken geometrically, the component of R along $HL = 0$, of $P = ab$, of $W = - ad$; also, along the other axis the compo-

nents of R, P, and W are respectively R, ac, $-ae$. Expressing these conditions trigonometrically, we have, first,

$$P \cos\beta - W \sin\alpha = 0,$$

or
$$P = \frac{W \sin\alpha}{\cos\beta}; \qquad (1)$$

and second, $R + P \sin\beta - W\cos\alpha = 0,$

or
$$R = W\cos\alpha - P\sin\beta.$$

Substituting the value of P from (1) in the preceding equation, we obtain

$$R = W\cos\alpha - \frac{W\sin\alpha\sin\beta}{\cos\beta},$$

$$= \frac{W(\cos\alpha\cos\beta - \sin\alpha\sin\beta)}{\cos\beta};$$

$$\therefore R = \frac{W\cos(\alpha+\beta)}{\cos\beta}. \qquad (2)$$

229. Special Cases. The values of P and R in terms of W and the angles α and β, derived in Art. 228, apply to all cases, whatever the direction of P. If now the power acts along the plane, or horizontally, these general equations take a special form applicable to the particular case.

(*a*) *The power acts along the plane* (Fig. 178). Here $\beta = 0$, and $\cos\beta = 1$; hence, from the general value

$P = \dfrac{W\sin\alpha}{\cos\beta}$, we obtain

$$P = W\sin\alpha, \qquad (3)$$

and, from the general value of

$R = \dfrac{W\cos(\alpha+\beta)}{\cos\beta}$, we obtain

$$R = W\cos\alpha. \qquad (4)$$

(*b*) *The power acts horizontally* (Fig. 179). Here $\beta = -\alpha$, $\cos(-\alpha) = \cos\alpha$. Hence, for

$P = \dfrac{W \sin\alpha}{\cos\beta}$, is obtained

$$P = \frac{W \sin\alpha}{\cos\alpha} = W \tan\alpha, \tag{5}$$

and, for

$R = \dfrac{W \cos(\alpha+\beta)}{\cos\beta}$, is obtained

$$R = \frac{W}{\cos\alpha} = W \sec\alpha. \tag{6}$$

From (5), if $\alpha = 90°$, $P = \infty$; that is, no finite force can support a body against a vertical surface if the surfaces in contact are perfectly smooth and there is no adhesion. This is only a special case of the general principle that the action of a force does not affect the motion of a body in a direction at right angles to that in which it acts.

230. The results in (*a*) and (*b*) of the preceding

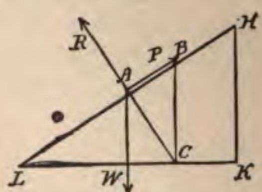

FIG. 178.

article can also be obtained independently by another method.

(*a*) *The power acts along the plane.* Let the lines P, R, W represent the three forces holding the body at

A in equilibrium. From P draw BC parallel to the direction of W; then the triangle ABC has its three sides respectively parallel to the three forces, and hence (132, *Cor.*) these sides are proportional to them. Again, the triangles ABC and KHL are mutually equiangular and similar, hence

$$P : W : R = AB : BC : AC,$$

$$= HK : HL : LK;$$

or
$$\frac{P}{W} = \frac{HK}{HL} = \sin\alpha, \qquad (1)$$

and
$$\frac{R}{W} = \frac{LK}{HL} = \cos\alpha. \qquad (2)$$

The result in (1) is sometimes stated in this form: *When the Power acts along the plane, the Power is to the Weight as the height of the plane is to the length.*

(*b*) *The power acts horizontally.* Let (Fig. 179) the

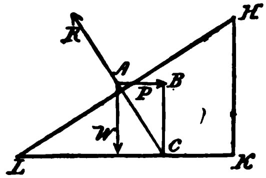

FIG. 179.

three forces P, R, W be represented by the three lines meeting at A. Through B draw BC parallel to the direction of W, and produce R to meet BC at C. The sides of the triangle ABC are respectively parallel to the three forces P, R, W, and therefore (132, *Cor.*) are pro-

plane? (*b*) acting horizontally? (*c*) acting at an angle of 30° with the plane?

2. What is the reaction of the plane in the three cases in example 1?

3. A force of 100 lbs. acts parallel to an inclined plane: What weight can it support in the following cases—the angle of the plane is (*a*) 10°, (*b*) 30°, (*c*) 45°, (*d*) 60°, (*e*) 80°, (*f*) 90°?

4. A weight of 100 lbs. rests on an inclined plane: What force acting parallel to the plane is required to support it, if the angle of the plane has the same values as in example 3?

5. A force of 100 lbs. acts horizontally to an inclined plane: What weight can it support in the different cases given in example 3?

6. A weight of 100 lbs. rests on an inclined plane: What force, acting horizontally, is required to support it in the several cases of example 3?

7. A railroad has a grade of 88 feet to the mile: What force must the locomotive exert to support the weight of the whole train, taking that at 25 tons?

8. The length, height, and base of a plane are in the ratio of 5 : 3 : 4. Into what two parts may a weight of 56 lbs. be divided so that one part resting on the plane may be supported by the other hanging over the top vertically downward?

9. If a horse can raise 600 lbs. vertically, what weight can he raise on a railway having a grade of 3°?

10. The grade of a railway is 44 feet to the mile: What power (acting parallel) is required to support any given weight?

11. A body is supported on an inclined plane by a force of 40 lbs. acting parallel to the plane; but if the force acts horizontally it must equal 50 lbs.: Required the weight of the body, and the inclination of the plane.

12. Two inclined planes of lengths 40 and 60 feet are placed so that they slope in opposite directions, and their equal heights, 12 feet each, coincide; a weight of 8 lbs. is supported in the longer plane by a string, parallel to the plane, passing over a pulley at the top, and attached to a second weight on the shorter plane: What is the second weight?

13. Weights of 8 and 12 lbs. are supported in equilibrium on two inclined planes, so placed that their equal heights of 6 feet

each coincide; they are attached to the extremities of the same string passing, parallel to the planes, over a pulley at the top: What are the lengths of the planes, the angle of the first plane being 30°?

VI. WEDGE.

233. The WEDGE in its simplest form is a five-sided solid, of which two adjacent sides meeting in the edge are rectangles, the two opposite ends are triangles, and the back is a rectangle. The power is supposed to act in a direction perpendicular to the back, and, assuming the surfaces in contact to be perfectly smooth, the resistances are felt in the same plane perpendicular to the sides.

234. Suppose the triangle in Fig. 181 to represent the section made by a perpendicular plane through the

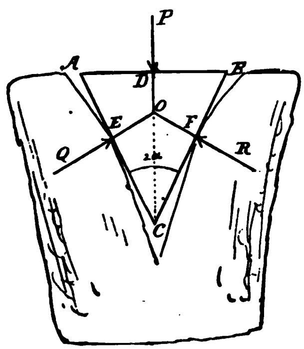

FIG. 181.

wedge; then, if the three forces P, Q, R hold the wedge in equilibrium, their lines of direction will, if produced, all meet at some point O. Also, since these forces are by supposition at right angles to the sides of the triangle

ABC, these sides will be respectively proportional to them; that is,

$$P : Q : R = AB : AC : BC.$$

If (as in Fig. 181) the section is an isosceles triangle, $AC = BC$, and $Q = R$; hence

$$\frac{P}{R} = \frac{AB}{AC}.$$

But $AB = 2AD$; and if $ACB = 2\alpha$, $AD = AC \sin \alpha$, or $AB = 2AC. \sin \alpha$; that is,

$$\frac{P}{R} = \frac{2AC \sin \alpha}{AC} = 2 \sin \alpha.$$

It appears from this equation that the mechanical advantage increases as the angle of the wedge diminishes.

235. Wedge on the Principle of Work. Let ABC (Fig. 182) be an isosceles wedge, and let the power acting against the resistances Q and R $(= 2R)$ force the wedge uniformly in a distance equal to DC. The work done by P is $P.DC$. The effective distance through which the resistance has been overcome is $D'E$. Therefore

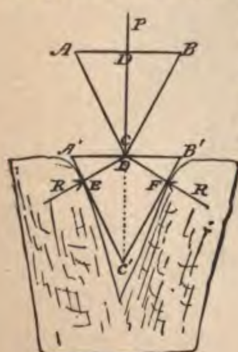

Fig. 182.

$$P.DC = 2R.D'E,$$

$$D'E = D'C' \sin \alpha;$$

$$\therefore P.DC = 2R.DC \sin \alpha,$$

or $$P = 2R \sin \alpha.$$

236. Application of the Wedge. In practice the relation established for the wedge has little value, for the

resistance due to friction is enormous. The wedge, however, is an important instrument; it appears in many cutting tools, such as the knife, chisel, axe, plane, and so on. It should be noticed that in the case of the plane, for example, if used for cutting soft wood, the angle is small and the edge sharp; for harder wood the angle is larger. The tool for planing iron has a very large angle—varying, say, from 60° to 80°.

When the wedge is employed as for cleaving wood, the resistance due to the cohesion and friction combined is so great that, instead of the pressure supposed in the above article, a blow from a heavy body, as an axe, is used to drive it in. In this the principle explained in Art. 105 is employed; the energy of a heavy body in motion being expended through a very small distance, and hence overcoming a great resistance. A nail is a familiar form of wedge, and its use further illustrates this principle; it depends upon friction for its hold in the substance into which it has been driven.

EXAMPLES.

XXXII. *Wedge.* Articles 233–235.

1. A wedge is isosceles in shape and has an angle of 20°; if $P = 40$ lbs., what is the resistance on each face?

2. A wedge is isosceles, and the angle 90°; a force of 100 lbs. acts at the back: What are the other two forces?

3. A wedge is isosceles, the power acting on the back is 40 lbs., and the forces on the other sides are 60 lbs. each: What is the angle of the wedge?

4. A wedge is isosceles and has an angle of 60°: What is the relation between the three forces?

5. The wedge is right-angled and the three sides have lengths of 15, 12, and 9 (back): If $P = 100$, what are the other two forces?

6. The angle of the wedge is 30°, the back is 10, and one side is 20: What is the ratio of the three forces?

VII. Screw.

237. The Screw consists of a solid cylinder with a raised portion passing spirally about it, which is called the thread. This thread is either rectangular or triangular in shape. It may be regarded as generated by the revolution of a rectangle, in the one case, or an isosceles triangle, in the other, about the cylinder at the same time that it advances uniformly parallel to the axis, and at such a rate that in each revolution it goes a distance equal to its own width. The two kinds of threads are shown in Figs. 183, 184.

The screw in use works in a nut whose parts are com-

Fig. 183.

Fig. 184.

plementary to those of the screw, so that the one fits closely into the other. The power acts at the end of a lever-arm to turn the screw in the nut; either one may be made stationary, so that the other moves with reference to it. The pressure of the weight or resistance is felt in the direction of the longer axis of the screw, but the resistance between the screw and nut is felt at each point of contact between them and perpendicular to their common surface.

238. *In the screw the Power is to the Weight as the distance between the threads is to the circumference described by the Power.*

Suppose the surface of the cylinder, about which the

thread passes as a spiral line (its thickness being neglected), to be unrolled on a plane. A rectangle with a series of equal triangles is the result, as shown in Fig. 185. Here AB is the circumference of the cylinder $= 2\pi r$, if r is the radius of the cylinder; CAB is the angle between the thread and a horizontal line, called the *angle of the screw;* BC is equal to the distance between the threads, also called the *pitch* of the screw.

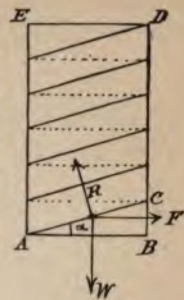

FIG. 185.

$$BC = AB \tan \alpha;$$

$$\therefore \text{distance between threads} = 2\pi r \tan \alpha.$$

Again, at each point of contact between the screw and the nut a force acts horizontally and holds both (1) P, acting on the lever-arm R (Fig. 186), and (2) W, acting vertically downward in equilibrium. Let the sum of these partial horizontal forces be represented by F; then, taking moments about the axis,

FIG. 186.

$$P.R = F.r. \qquad (1)$$

For P and F tend to turn the cylinder in opposite directions. Also, on the principle of the inclined plane, since the force F acting horizontally [229 (5)] supports the total weight W,

$$F = W.\tan \alpha. \qquad (2)$$

From (1) and (2)

$$P.R = W.r \tan \alpha,$$

$$\frac{P}{W} = \frac{r \tan \alpha}{R};$$

multiplying by 2π,

$$\frac{P}{W} = \frac{2\pi r \tan \alpha}{2\pi R}.$$

But $2\pi r \tan \alpha =$ distance between the threads, and $2\pi R$ is the circumference described by the power-arm; hence the relation already given is deduced.

239. Screw on the Principle of Work. Let the power acting on its lever-arm cause it to describe a complete circumference $2\pi R$; the work done by the power is then $P.2\pi R$. At the same time the weight has been raised (or resistance overcome) through a distance equal to that between two consecutive threads. Therefore

$$P.2\pi R = W \times \text{dist. between threads},$$

$$\frac{P}{W} = \frac{\text{dist. between threads}}{2\pi R}.$$

240. Application of the Screw. The screw is practically a most important mechanical instrument, being used in many cases where a great weight is to be raised or a heavy pressure to be exerted. For example, buildings are often raised by the combined use of a number of screws, and screw-presses are employed for many different purposes.

By increasing the length of the power-arm, or diminishing the distance between the threads, any required mechanical advantage may *theoretically* be obtained ; in fact, however, a limit is soon reached; friction is a

serious element, and the *modulus* of the machine is small. The use of the common screw, as, for example, in binding two boards together, depends for its efficiency entirely upon friction.

241. Micrometer Screw. The screw is also employed for measuring very small distances, and is then called a *micrometer screw*. In this case the relation in velocity of motion of the parts is the matter considered, and the relation of P to W is lost sight of. The principle of the micrometer screw will be clear from the following remarks. Suppose a screw with 100 threads to the inch: it is obvious that each complete revolution will advance the screw if the nut is stationary, or the nut if the screw is held firm, through a distance of $\frac{1}{100}$ of an inch. Suppose, further, that the head of the screw is a circle whose circumference is graduated into 100 equal parts: then, if it is arranged with a fixed index, it is easy to turn the head—that is, the screw—through $\frac{1}{100}$ of a revolution, and this will cause an advance of $\frac{1}{100}$ of $\frac{1}{100}$ of an inch; that is, $\frac{1}{10000}$ of an inch for the screw itself. Screws with very fine threads, and hence giving very slow motion, find many applications in physical apparatus.

242. Differential Screw. The *differential screw* gives a greater mechanical advantage, and hence a slower motion (which may be the end desired), than can be conveniently obtained from the simple form. Here a larger screw turns in a fixed nut, and a smaller one with a less pitch turns inside of it. The power acts directly on the larger screw, and the resistance is felt by the smaller. Hence it is evident that while the large screw descends the small screw ascends, and the actual motion

of the platform is the resultant of these two opposite velocities.

The relation of P to W is given immediately by the principle of work. Suppose the power to act through one circumference $(2\pi R)$; the larger screw will descend a distance equal to the distance between its threads (p), and the smaller screw will ascend the distance between its threads (p'); hence the difference of these two will represent the distance through which weight is raised (or resistance moved).

$$\frac{P}{W}=\frac{h}{s}=\frac{p-p'}{2\pi R}.$$

The Power is to the Weight as the difference of the distances between the threads of the two screws is to the circumference described by the power-arm.

243. Endless Screw. Fig. 187 represents what is called

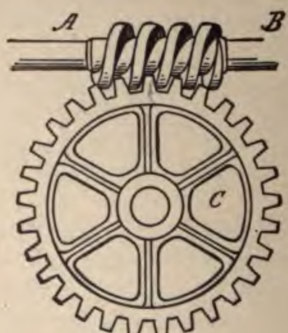

FIG. 187

an *endless screw*. Here there is a cylinder with a uniform thread fitting into the teeth of a toothed wheel. The number of threads in the screw of the cylinder AB

will determine the number of teeth of the wheel *C*, which will be advanced in one revolution of the former. Upon this depends the mechanical advantage of the arrangement.

EXAMPLES.

XXXIII. *Screw.* Articles 236–242.

1. What weight can be raised by a power of 30 lbs. acting on a lever-arm 2 feet long, if the screw has 2 threads to the inch?

2. If a power of 40 lbs. acting on an arm 25 inches long can support a weight of 8000 lbs., what will be the distance between the threads of the screw?

3. A screw has 10 threads to the inch; the circumference described by the power is 4 feet: What power is needed to support a weight of 6000 lbs?

4. While the point of application of the power makes a revolution of 3 feet, the screw advances $\frac{1}{4}$ of an inch; the power is 50 lbs.: What is the weight raised?

5. The angle of the screw is $10°$, and the length of the power-arm is twenty times the radius of the cylinder: What is the mechanical advantage?

6. The circumference described by the power-arm is 20 feet, and the mechanical advantage 480: How many threads in the screw are there to the inch?

7. The circumference described by the power-arm is 14 feet, the power is 60 lbs., and the weight 6 tons (6×2240 lbs.): What is the distance between the threads of the screw?

8. The power-arm of a differential screw is 18 inches; there are 6 threads to the inch in the larger screw, and 8 threads in the smaller; the power is 30 lbs.: What weight can be supported?

CHAPTER IX.—PENDULUM.

244. Motion in a Vertical Circle. **Let ABC (Fig. 188) represent a vertical circle, regarded as perfectly smooth. Suppose a particle to start from rest at A and slide down**

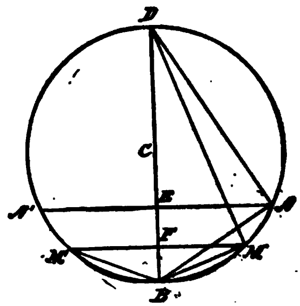

FIG. 188.

toward B; its velocity (v) at any point M will be the same (40) as if it had fallen through the vertical height EF; that is,

$$v^2 = 2g.EF. \qquad (1)$$

Since DMB and DAB are right angles, by geometry,

$$\overline{MB}^2 = DB.FB, \qquad (2)$$

$$\overline{AB}^2 = DB.EB. \qquad (3)$$

Let $AB = a$, and $MB = z$; also, $DB = 2r$; substituting these values and subtracting (2) from (3),

$$a^2 - z^2 = 2r\,(EB - FB) = 2r.EF. \qquad (4)$$

Introducing the value of EF from (4) in (1), we obtain

$$v^2 = \frac{g}{r}(a^2 - z^2), \qquad v = \pm\sqrt{\frac{g}{r}(a^2 - z^2)}. \quad (5)$$

For the point A (or A'), $z = a$, and therefore $v = 0$; for B, $z = 0$ and $v = \pm a\sqrt{\frac{g}{r}}$, the double sign indicating that the motion may be from A toward $A'(+)$, or the reverse $(-)$. For two points M and M', equally distant from B, $BM = +z$, and $BM' = -z$; for both these the value of v is the same, and for each there is a $+$ value and a $-$ value, according to the direction of the motion. It is evident that if the particle were projected from B with a velocity equal to $a\sqrt{\frac{g}{r}}$, it would ascend to A' before coming to rest.

245. Motion of a Simple Pendulum. An ideal simple pendulum is a material particle attached by a string without weight to the point of suspension, and vibrating without resistance from friction or any other source. A particle so suspended will, if set in motion, continue to vibrate to and fro in an arc of a circle, and will follow the same laws as the body descending the smooth curve considered in the previous article. The tension of the string, like the resistance of the plane, is always equal to one component of the weight, and is in both cases exerted at right angles to the direction of motion, and hence does not influence the velocity of the particle.

Therefore the value of the velocity, $v = \sqrt{\frac{g}{r}(a^2 - z^2)}$, obtained in Art. 244 will apply also to the pendulum at

any point in its course. If now the radius r (the length of the pendulum) is great and the length of the arc of vibration is very small, we may take a and z in this value of v as representing the *arcs* instead of the *chords*, and this is assumed in the following demonstration. The error arising from this assumption may be neglected without destroying the value of the result.

Let aa' (Fig. 189) represent the arc of vibration of the simple pendulum, whose length (r) is CB. Take the straight line AA' equal to aa', and so that every point m on the arc has a corresponding point M on the straight line. We may without error imagine the pendulum to vibrate from A to A', following the same law as in the arc aa', so that its velocity at any point M will be expressed by the above formula,

$$v = \sqrt{\frac{g}{r}(a^2 - z^2)},$$

when $AB = BA' = a$, and $BM = z$. Here, as remarked above, a and z are arcs, not chords.

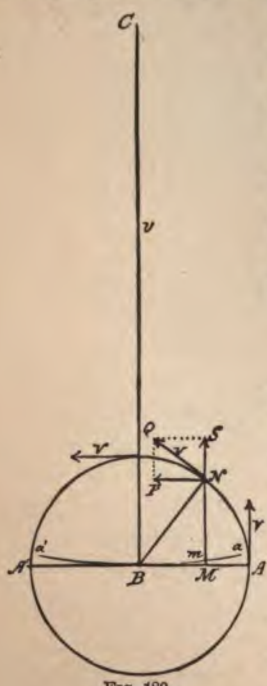

FIG. 189.

Upon AA' as a diameter describe the circle ANA', and suppose a particle to move *uniformly* about the semi-circumference ANA' with the constant velocity $a\sqrt{\frac{g}{r}}$. Then, since the distance passed over is πa, the time (t) required for this imaginary particle to go from A to

A', since the distance and velocity are known, is (19) as follows:

$$t = \frac{s}{v} = \frac{\pi a}{a\sqrt{\frac{g}{r}}} = \pi\sqrt{\frac{r}{g}}.$$

It will now be shown that this expression also gives the time required by the pendulum to vibrate from A to A' (that is, from a to a').

The constant velocity V of the imaginary particle, viz. $a\sqrt{\frac{g}{r}}$, may be represented (20) at any point in its path, as N, by a tangent to the curve NQ $\left(= a\sqrt{\frac{g}{r}}\right)$. The horizontal component of this velocity is then represented by NP.

$$NP = NQ.\cos QNP,$$

$$= a\sqrt{\frac{g}{r}}.\cos BNM.$$

But $a \cos BNM = NM$, and

$$NM = \sqrt{\overline{NB}^2 - \overline{BM}^2} = \sqrt{a^2 - z^2}.$$

Therefore $$NP = \sqrt{\frac{g}{r}(a^2 - z^2)}.$$

Now, this expression for the horizontal component of the velocity of the imaginary particle at N is also the value of the velocity of the pendulum at the corresponding point M. The meaning of this result is as follows: If, at the same instant that the pendulum

through these particles and parallel to the axis of suspension is called the *axis of oscillation.* The distance between these two axes is the length of the compound pendulum as above defined.

247. To find the axis of oscillation, use is made of the principle: *The axis of suspension and axis of oscillation are interchangeable.* Therefore, if the pendulum be swung on one axis and the time of vibration be determined, and then the second axis be found, so that if this is made the axis of suspension the time of vibration will be exactly the same, the latter axis is the axis of oscillation. The truth of this principle is established in works on higher Mechanics.

The pendulum exhibited in Fig. 190 is a form devised by Kater: *a* and *b* are the two axes of suspension and oscillation; *v* and *w* are two slides, the position of which may be adjusted until the condition in regard to the equal times of vibration for the two axes is satisfied.

FIG. 190.

248. Application of the Pendulum. The most important application of the pendulum is as an instrument for determining the value of the acceleration of gravity (g). The direct determination of g requires the observation of either the velocity acquired (32 feet per second in a second) or the space passed over (16 feet in the first second from rest) by a body (27) falling in a vacuum. This method is obviously impracticable. If Attwood's machine (74) be employed, the force of gravity may, as it were, be weakened so that the velocity acquired and space passed through are much less than for a body falling freely. In this way approximate values of g may be obtained.

For accuracy, however, the seconds pendulum gives the simplest and most satisfactory method of determining the value of g. From Art. 245 we see that

$$t = \pi\sqrt{\frac{l}{g}}, \quad \text{or} \quad g = \frac{\pi^2 l}{t^2}.$$

For the seconds pendulum, $g = \pi^2 l.$

In the actual determination of the value of l for a pendulum vibrating in seconds on any point of the earth's surface, many refinements of observation are required. It is sufficient, however, to say here that when the experiments are carried on with all possible care, and when the many necessary corrections have been introduced, a very high degree of accuracy is obtainable for the value of l, and consequently of g.

249. The following table gives the lengths of the seconds pendulum and the value of g (for the sea-level) for some important points on the earth. Those for the equator and pole are calculated from an equation the constants of which have been deduced from numerous pendulum experiments in different localities; the others are from direct observations (U. S. Coast Survey):

	Latitude.	Value of g, in feet-per-second per second.	Value of l, in inches.
Equator..................	0°	32.091	39.017
New York (Hoboken)......	40° 43′	32.161	39.103
Paris......................	48° 50′	32.185	39.133
London (Kew)............	51° 29′	32.193	39.142
~~Berlin....................~~	~~52° 30′~~	~~32.195~~	~~39.144~~
Pole.......................	90° 0′	32.255	39.217

250. The lengths of the seconds pendulum, as determined by numerous experiments, are of great value, as giving the means of comparing the intensity of the force

that

of gravity at the different stations. By this means the ellipticity of the earth has been determined. A formula, alluded to in the previous article, has also been obtained which gives the values of l and g for any required latitude. The ellipticity of the earth, obtained in this way, differs somewhat from the similar result from the trigonometrical measurements of long arcs of meridians. Moreover, the values of l and g derived from the formula do not always agree with those obtained by direct experiment as closely as might be expected. The explanation of the variations is to be found in the facts (1) that the earth is rather an ellipsoid than a spheroid, and, further (2), that the density of the earth's crust at different points is not uniform. This latter point has been extensively investigated by means of these pendulum observations; it is found, for example, that the attraction on the coast is greater than in the interior, and that it is still greater on islands in the sea. From this it is argued that the density of the earth's crust under the ocean is greater than the average of that forming the dry land. Other similar results have also been obtained by this means.

251. It has also been proposed to make the pendulum a basis of a system of weights and measures, on the ground that it would, at a given place, be a standard which could be at any time replaced if others were destroyed. It was, for example, enacted by Parliament in 1824 that the length of the standard yard should bear to the length of the seconds pendulum in London (in vacuum and at the sea-level) the ratio of 36 : 39.1393. The difficulty in obtaining the length of the pendulum with the degree of accuracy now needed for a standard of measures makes the relation of little practical value

The pendulum finds a further application as a regulator for clocks.

EXAMPLES.

XXXIV. *Pendulum.* Articles 244–247.

[The length of the seconds pendulum at New York is about 39.10 inches, and $g = 32.16$.]

1. Required the length of a pendulum at New York which will vibrate (*a*) in $\frac{1}{2}$ second, (*b*) in 2 seconds, (*c*) in $2\frac{1}{2}$ seconds.

2. Required the length of the seconds pendulum where the acceleration of gravity is 32.25 (that is, near the pole).

3. Required the length of a pendulum to vibrate in 2 seconds, where the value of g is 32.1.

4. What would be the length of a pendulum to vibrate in 1 second at the surface of the sun (acceleration of gravity $= 28\,g$)?

5. How many beats in a minute would a pendulum 8 feet long make in New York?

6. If a pendulum 24 inches long vibrates in $\frac{4}{5}$ second, what is the length of a seconds pendulum?

ADDITIONAL EXAMPLES,

INTRODUCING THE METRIC UNITS.

I. *Uniform Motion of Translation or Rotation.* Articles 17–21; pp. 9–12.

1. A body travels 10 meters per second: How far will it go in a day of 24 hours?

2. A velocity of 50 kilometers per hour corresponds to a rate of how many meters per second?

3. A man walks uniformly 6 kilometers per hour: (*a*) How many decimeters does he go in a second? (*b*) How many meters in a minute?

4. Two bodies start from the same point in opposite directions; the one moves at a rate of 3 meters per second, the other at a rate of 30 kilometers per hour: (*a*) What will be the distance between them at the end of 8 minutes? (*b*) When will they be 6 kilometers apart?

5. How far will the bodies in the preceding example be apart at the end of the same time, if they move in the same direction?

6. (*a*) What is the angular velocity of a revolving wheel having a radius of 1 meter, if it makes 14 revolutions per second (take $\pi = 3\frac{1}{7}$)? (*b*) How far (in kilometers) will a point on the circumference travel in 10 hours?

II. *Uniformly Accelerated Motion* Articles 22–28; pp. 13–20.
A. Falling Bodies ($g =$ about 9.80 meters at New York).

[The body is supposed to start from rest.]

1. Calculate the distances fallen through (in meters) and the acquired velocities in 1, 2, 3, 4 seconds from rest.

2. A body falls 10 seconds: Required (*a*) the velocity acquired; (*b*) the whole distance fallen through; (*c*) the space passed over in the last second of its fall.

3. A body has fallen through 90 meters: Required (*a*) the time of falling; (*b*) the final velocity.

4. A body has acquired in falling a velocity of 73.5 meters per second: Required (*a*) the time of falling; (*b*) the distance fallen through.

5. A body in falling passed over 44.1 meters in the last second: Required (*a*) the time of falling; (*b*) the distance fallen.

B. General Case.—Acceleration $= f$.

1. A body moves 100 meters in the first 5 seconds from rest: What is the acceleration?

2. A body moves 10 meters in the first second: (*a*) What is the acceleration? (*b*) How far will it go in 6 seconds? (*c*) What will be its final velocity at the end of this time?

3. The acceleration is 12 meters-per-second per second: (*a*) What velocity does a body acquire in 5 seconds? (*b*) What space does it pass over?

4. A body moves 54 meters in 3 seconds, and 96 in the next 2: Is its motion uniformly accelerated?

5. A body passes over 64 meters in 4 seconds: What distance must it go in the next 5 to satisfy the condition of uniformly accelerated motion?

6. The velocity of a body is increased from 20 to 40 meters per second while it passes over 30 meters: What is the acceleration?

III. *Composition of Constant Velocities.* Articles 29–37; pp. 22–27.

1. The velocity of a steamboat is 10 kilometers per hour, that of the stream is 8, and a man walks the deck from stern to bow at the rate of 6: Required the actual velocity of the boat (*a*) if headed up stream, and (*b*) down stream; also (*c*, *d*), that of the man in each case.

2. The velocities of the boat and stream are respectively 100 meters and 80 meters per minute, and the boat is headed directly across the stream (Fig. 11): (*a*) What will be the actual direction of the boat's motion? (*b*) What the rate of its motion? (*c*) How long will the passage take if the stream is 2 kilometers in width?

3. The velocities of boat and stream are as in 2, but it is required that the boat shall go directly across from A to C (Fig. 12): (*a*) In what direction must the boat be headed? (*b*) What will be its actual velocity across? (*c*) What time will the passage take, the width being as in 2?

4. A ball on a horizontal surface tends to move north with a velocity of 12 meters per second, and east with a velocity of 5 meters per second: (*a*) What will be the actual velocity, and (*b*) in what direction?

5. A ball, moving north at a rate of 8 meters per second, receives an impulse tending to make it move due south-east with the same velocity: (*a*) What path will it take, and (*b*) at what rate will it move?

6. A man, skating uniformly at a rate of 4 meters per second, projects a ball on the ice in a direction at right angles to his motion at a rate of 3 meters per second: What is (*a*) the actual rate, and (*b*) the direction of its motion (friction neglected)?

IV. *Resolution of Constant Velocities.* Article 38; pp. 28, 29.

1. A ball tends to move in a certain direction at a rate of 6 meters per second, but it is constrained to move at an angle of 60° with this direction: Required its velocity in the latter direction.

2. A ball rolls at the rate of 6 meters per second across the diagonal of a rectangular room $ABCD$ whose dimensions are $15 \times 20\ (= AB \times AC)$: What is its rate of motion parallel to each side?

3. A body moves N. 30° E. at a rate of 5 kilometers per hour: Required its rate of motion northerly and easterly.

4. A boat, though headed directly across a stream, actually moves diagonally across the stream at an angle of 30° (BAC, Fig. 11) and at a rate of 15 kilometers per hour: Required (*a*) the rate of the boat, and (*b*) of the current taken independently.

V. *Falling down an Inclined Plane.* Article 40; pp. 32, 33.

[The plane is supposed to be perfectly smooth, so that there is no friction.]

1. The angle of the plane is 30°: Required (*a*) the acceleration

down the plane; (*b*) the distance fallen through in 4 seconds; (*c*) the velocity acquired.

2. The height of the plane is 19.6 meters and the length 78.4: (*a*) What is the time required to reach the bottom? (*b*) What is the velocity acquired?

3. The angle of the plane is 45°: Required the time of falling 490 meters.

4. The length of a plane is 630 meters; a body falls down it in 30 seconds: (*a*) What is the acceleration? (*b*) What is the height of the plane?

VI. *Bodies projected vertically downward.* Articles 41, 42; pp. 34, 35.

1. A body is thrown vertically down with an initial velocity of 12 meters per second: Required (*a*) the velocity at the end of 7 seconds; (*b*) the distance fallen through.

2. A body is projected down with an initial velocity of 19.1 meters per second: (*a*) How long will it require to fall 218 meters? (*b*) What velocity will it then have?

3. What velocity of projection must a stone have to reach the bottom of a cliff 100 meters high in 3 seconds?

4. A stone is dropped from a bucket which is descending a shaft at the uniform rate of 3.5 meters per second, and at the moment when the bucket is 75 meters from the bottom: (*a*) How far will they be apart in two seconds? (*b*) When will the stone reach the bottom?

VII. *Bodies projected vertically upward.* Article 44; pp. 37–39.

1. The velocity of the projection upward is 49 meters per second: Required (*a*) the time of ascent; (*b*) of descent; (*c*) the height of ascent; (*d*) the distance gone in the first and last seconds of ascent.

2. A body is projected up with a velocity of 42 meters per second: (*a*) When will it be 37.1 meters above the starting-point?

3. What velocity of projection must a ball have in order to ascend just 160 meters?

4. What time does a body require to ascend 250 meters, that being the highest point reached?

5. A ball thrown up passes a staging 36.4 meters from the ground at the end of 2 seconds: (*a*) What was the velocity of the projection? (*b*) When will it pass it again?

VIII. *Projected up or down a smooth Inclined Plane.* Article 45; p. 39.

1. The height of the plane is 105 meters, the length is 420 meters, the velocity of projection down is 20.3 meters per second: (*a*) How long will it require to descend? (*b*) What will be the final velocity?

2. The angle of the plane is 30°, the velocity of projection down is 10 meters per second: Required (*a*) the velocity at the end of 4 seconds; (*b*) the distance gone through.

3. The height and length of the plane are 160 and 320 meters respectively: (*a*) What velocity is required that the body should just reach the top? (*b*) What time is needed?

IX. *Bodies projected against Friction.* Articles, 41, 42; pp. 34, 35.

[The retardation (or minus acceleration) due to friction takes the place of the f in the formulas of articles 42 and 44.]

1. A body projected on a *rough* horizontal plane has at starting a velocity of 40 meters per second, but loses this at the rate of 4 meters for each succeeding second: (*a*) What is the retardation (minus acceleration) due to friction? (*b*) When will the body stop? (*c*) How far will it have gone?

2. The retardation due to friction is for each second 6 decimeters per second for a given sliding body, the initial velocity is 12 meters per second: Required (*a*) the time it will continue to slide; (*b*) the distance it will go; (*c*) its velocity at the end of 3 seconds.

3. If the retardation of friction is 2 decimeters-per-second per second: (*a*) What initial velocity (in kilometers per hour) must a body have in order to slide just 160 meters? (*b*) If the velocity is doubled, how much farther will it go?

X. *Projectiles.* Articles 47–51; pp. 43–50.

1. The initial velocity of a projectile is 245 meters per second, and the angle of elevation is 30°: Required (*a*) the time of flight; (*b*) the range.

2. The initial velocity is 140 meters per second: What angle of elevation will give a range of 1.5 kilometers? Show that there are two answers.

3. The angle of elevation is 15°: What initial velocity is required that the range should be 4 kilometers?

4. A rifle-ball is shot horizontally from the top of a tower 44.1 meters high, and with an initial velocity of 400 meters per second: When and how far from the base of the tower will it strike the horizontal plane below?

5. A ball is thrown horizontally from the top of a cliff above the sea; it strikes the water in 5 seconds, and at a horizontal distance of a kilometer: What was (*a*) the initial velocity, and (*b*) what was the height of the cliff?

XI. *Mass—Density—Volume.* Article 56; pp. 53, 54.

1. What is the ratio in volume of a piece of silver weighing 20 kilograms and having a density of 10.5 (referred to water as unity), and a piece of iron weighing 5 kilograms and having a density of 7 ?

2. What is the ratio in weight (that is, in mass) of two blocks of stone, one having a volume of 1 cubic decimeter and a density of 3, the other a volume of 400 cubic centimeters and a density of 2.75 ?

3. If a liter of dry air at 0° weighs 1.2932 grams, and a liter of water at the same temperature weighs 999.88 grams, what is the density of the water referred to that of air as unity ?

4. What is the weight of a liter of mercury at 100°, the expansion of volume from 0° to 100° being in the ratio of 1 : 1.0154 ? The density of mercury at 0° is 13.6, referred to water as unity.

XII. *Force of Gravity.* Articles 63–65; pp. 58–62.

1. At what distance from the centre of the earth would a mass of matter weighing 16 kilograms on the earth's surface exert a full equivalent to 1 kilogram on a spring-balance ?

2. If the mass of the sun is 355,000 times that of the earth, and its diameter 112 times, what would be the acceleration of gravity at its surface ?

3 If the moon's mass is $\frac{1}{80}$ of that of the earth, and its dia-

meter 3476 kilometers, that of the earth being about 12,715 kilometers, what is the acceleration of gravity on the moon's surface?

4. If the distance from the earth to the moon is 60 times the earth's radius, what is the force of the earth's attraction at the moon?

XIII. *Collision of Inelastic Bodies.* Article 70; pp. 70, 71.

[The bodies are supposed to be perfectly inelastic, their motion is uniform, and the impact is direct.]

1. A ball weighing 10 kilos and having a velocity of 6 meters per second overtakes a second ball weighing 5 kilos and whose velocity is 3 meters per second: What is the final velocity?

2. If the first ball in the preceding example meets the second, what is the final velocity?

3. A body weighing 40 kilos strikes another at rest weighing 360 kilos, and the two move on with a velocity of 1 meter per second: What was the original velocity of the first ball?

4. Three bodies, each weighing 4 kilos, are situated in a straight line; a fourth, weighing 8 kilos and moving at a rate of 6 meters per second, strikes them in succession: What velocity results after each impact?

5. A rifle-bullet weighing 30 grams is fired into a suspended block weighing 15 kilos; the blow causes the wood to rise 36 millimeters: Required the velocity of the bullet at the moment of impact.

XIV. *General Dynamical Problems.* Articles 68, 73–76; pp. 66, 67, and 75–80.

1. If in Attwood's machine $P = 102$ grams and $Q = 45$ grams: (*a*) What is the acceleration? (*b*) What space will be passed through in 2 seconds?

2. (*a*) At what height above the earth's surface would a body fall 625 millimeters in the first second from rest? (*b*) If its weight was 16 kilos, what pull would it exert on a spring-balance at this point?

3. A weight of 6 kilos hanging over the edge of a smooth table drags a weight of 15 kilos with it: What is the acceleration and the tension of the string?

4. For what time must a force of 60 grams (gravitation measure) act on a body weighing 2 kilos to give it a velocity of 6 metres per second?

5. A body weighing 140 kilos is moved by a constant force, which generates a velocity of 2 meters per second in one second: What weight could the force support?

6. What constant force (*a*) in gravitation measure (grams), (*b*) in absolute measure (dynes), will cause a body weighing 490 grams to pass over 400 meters in 10 seconds on a smooth horizontal surface?

7. A constant force of 980 dynes gives a body an acceleration of 7 meters per second in one second: What is the weight of the body (in grams)?

8. What weight could a force equal to 1 dyne support?

XV. *Centripetal and Centrifugal Forces.* Articles 77–81; pp. 83–8.

1. A ball weighing 10 kilos is whirled by means of a string around a centre at a radius of 2 meters, with a linear velocity of 7 meters per second: What is the value of f, and what is the tension of the string (F)?

2. A ball weighing 14 kilos attached to a centre at a distance of 3 meters makes 420 revolutions in a minute ($\pi = 3\frac{1}{7}$): What is the pull on the centre?

XVI. *Friction.* Articles 82–94; pp. 90–98.

1. A force of 6 kilos is just sufficient to move a body weighing 48 kilos uniformly along a horizontal plane: What is the coefficient of friction?

2. The value of μ is .3, the weight of the body is 16 kilos: What force is required to move it uniformly?

3. It is found that a force of 40 grams suffices to move a body uniformly on a horizontal surface, where the value of the coefficient of friction is known to be .25: What is the weight of the body?

4. A body weighing 15 kilos is just on the point of sliding when the surface it rests upon is inclined 20°: (*a*) What is the coefficient of friction and the force of friction? (*b*) If the weight of the body is doubled, what values have these quantities?

5. A body weighing 12 kilos rests on an inclined plane whose angle of inclination is 30° and where $\mu = .6$: What is the force of friction?

XVII. *Work.* Articles 95–100; pp. 101–105.

[The UNIT OF WORK is usually taken as one *kilogram-meter;* on the C. G. S. system it is an *erg*, or one *dyne-centimeter.*]

1. How many foot-pounds correspond to one kilogram-meter?

2. A weight of 300 kilos is raised to the top of an inclined plane whose length is 1200 metres, and the angle of inclination = 10°: What work is done?

3. A sled weighing 600 kilos is dragged 15 kilometers on the snow, where the coefficient of friction is .075: What work is done against friction ?

4. How much work is done against friction in dragging a weight of 200 kilos a distance of 1000 meters along a horizontal plane, if the coefficient of friction is .5?

5. A weight of 100 kilos is dragged up an inclined plane whose length is 2600 meters, and whose height is 1000 meters ($\mu = .3$). How much work is done?

XVIII. *Potential and Kinetic Energy.* Articles 101–118; pp. 106–124.

1. How many kilogram-meters of work are equivalent to one heat-unit (*i.e.*, to raise 1 kilo of water 1° C.)?

2. The weights of a clock weigh 20 kilos, and they have 10 meters to fall· How much work do they represent when wound up ?

3. A mill-pond has a surface of 1000 square meters and an average depth of 1 meter; supposing it 50 meters above the sea-level, how much potential energy does it represent?

4. How much work is accumulated or stored up (= kinetic energy) in a cannon-ball weighing 100 kilos and moving at a rate of 280 meters per second? How much heat will be generated if its mass motion is entirely destroyed by the impact with the target?

5. A bullet weighing 30 grams has a velocity of 420 meters per second: How much work can it do?

6. A body weighing 12 kilos is projected along a rough horizontal plane ($\mu = .25$) with an initial velocity of 140 meters per second: How far (*a*) will it go before coming to rest, and how long (*b*) will it slide?

7. A hammer weighing 5 kilos and moving with a velocity of 1.4 meters per second drives a nail into a plank 1 centimeter: What resistance does it overcome?

8. A weight of 500 kilos, used as a pile-driver, falls 6 meters and drives the pile in 2 centimeters: What resistance does it overcome?

XIX. *Parallelogram of Forces.* Articles 124–136; pp. 129–142.

1. Two forces, $P = 70$ grams, $Q = 240$ grams, act at right angles to each other: Required the magnitude and the direction of their resultant.

2. Two forces, $P = 12$ kilos, $Q = 10$ kilos, act at an angle of 120°: Required the magnitude of R.

3. Of two forces, $P = 12$ and $Q = 24$ kilos, the angle between Q and R is 30°: Required R and the angle between P and Q.

4. A peg in a wall is pulled by two strings with forces of 4 kilos each; they are equally inclined downward (40°) to the vertical: What weight hung on the peg would give an equal strain?

5. A peg in a wall is pulled by two strings, one horizontal with a tension of 350 grams, and the other vertical with a tension of 840 grams: What single force would exert an equal pull upon it?

6. A weight is supported by two equal strings attached to nails in the ceiling and enclosing an angle of 120°; the tension of each string is 8 kilos: What is the weight supported?

XX. *Resolution of Forces.* Articles 137, 138; pp. 143–146.

1. A force of 60 kilos is exerted in a direction N. 20° E.: What portion of it is felt north? what portion east?

2. A weight of 12 kilos is supported by two strings at an angle of 120°; one (*a*) goes (Fig. 66, p. 144) horizontally to the vertical wall, and the other (*b*) to the ceiling: What is the tension of the two strings?

3. A picture, whose weight is 40 kilos, is supported by a cord

attached to the upper corners and carried over a nail so as to include an angle of 75°. If the top of the picture is horizontal, what are the tensions of the strings?

4. A horse drags a sled by a rope inclined at the ground at an angle of 10°; the tension of the rope is 300 kilos: What is the effective component of the force exerted?

5. A weight of 640 grams is supported by two strings, one of which makes an angle of 30° with the vertical, and the other 60°: Find the tension of each string.

XXI. *Resolution of Forces along two Rectangular Axes.*

Articles 140, 141, 147–149; pp. 147–149, 156–158.

1. Find the magnitude and direction of the resultant of the following forces: $P = 100$ kilos, $Q = 50$, $S = 200$. The angle between P and $Q = 60°$, between Q and $S = 270°$.

2. Required the magnitude and direction of the resultant of the following forces: $P = Q = S = T = 100$ kilos. The angles are as follows: between P and $Q = 60°$, between Q and $S = 120°$, between S and $T = 120°$.

XXII. *Parallel Forces.* Articles 143–149; pp. 152–159.

1. A rigid rod, supported at the ends A and B, has a weight of 24 kilos hung 2 meters from A and 4 meters from B: What pressures do the supports feel? The weight of the rod itself is neglected here, as, too, in the following examples.

2. ABC is a rigid rod; at B a weight W is hung, so that $AB = 24$ and $BC = 32$; the pressure at A is 14 kilos: What is the pressure at C, and what is W?

3. A weight of 80 kilos is carried by means of a rigid rod on the shoulders (at the same height) of two men A and B; the distances from them are 2 and 3 meters respectively: What weight does each carry?

4. A table has as its top an equilateral triangle ABC (Fig. 82); a weight of 10 kilos is placed at O, so that the perpendicular distance from O on $BC = 18$ centimeters, and those on AC, AB each equal 36 centimeters: What is the pressure on each of the three legs?

5 A rod 40 centimeters long and whose weight acts at its

middle point rests on two vertical props placed at the ends: Where must a weight, equal to twice that of the rod, be placed that the pressure on the props shall be as 4 : 1?

XXIII. *Moments.* Articles 151–156; pp. 161–167.

1. A force, $P = 24$ kilos, acts at right angles to an arm 3 meters long: What is its moment?

2. A rigid rod AB, 2 meters long and free to turn about B, is acted on by a force, $P = 30$ kilos, whose direction makes an angle of 30° with AB: What is the moment of P? If the angle is 150°, what is the moment?

3. A force, $P = 100$ kilos, acts at the extremity of a rod, AB, 4 meters long, and at an angle of 120°: What is the moment of P about B?

4. A bar 5 meters long and pivoted at the middle has a weight of 10 kilos hung at one extremity: What is the moment of the weight (a) when the bar is horizontal, (b) when it makes an angle of 20° below, and (c) of 70° above with the horizontal position?

XXIV., XXV. *Centre of Gravity—Stability.* Articles 159–177; pp. 169–189.

1. Where is the centre of gravity of two bodies, A and B, weighing 30 and 42 grams respectively, rigidly connected by a weightless rod 36 centimeters long?

2. A rod AB, 1 meter long and weighing 250 grams, has a weight $P = 4\frac{3}{4}$ kilos hung at the end B: Where will it balance?

3. What weight must be hung at the end of a rod $\frac{1}{2}$ meter long and weighing 30 grams that it may balance 25 millimeters from that end?

4. A rod 2 meters long and having a weight of 5 kilos at one end balances at a point 2 millimeters from this end: What is its weight?

5. A uniform rod AB, 1 meter long and weighing 1 kilo, has a weight of 750 grams at the end A, and one of 250 at the end B: Where will it balance?

6. A uniform metal wire is bent into the form of an isosceles triangle, ABC, so that $AB = AC = 117$ millimeters, and $BC = 90$ millimeters: Where is the centre of gravity?

7. A table 2 meters square stands upon four legs, each of which is 300 millimeters in from the adjacent edges; its height is 800 millimeters and its weight 24 kilos: What is the least force required to put it on the point of overturning if applied at the edge (*a*) as a horizontal push? (*b*) as a pressure directly down?

8. A table, having a circular top of $\frac{1}{2}$ meter radius, is supported on four legs placed at the edge and at equal distances from one another; the height is $\frac{3}{4}$ meter and the weight 16 kilos: What is the least force that will put it on the point of turning it if applied at the top (*a*) as a horizontal push? (*b*) as a pressure down? (*c*) acting vertically upward?

9. What work would be done in overturning a cylindrical column of stone weighing 10,000 kilos, 3 meters high and 1 meter in diameter, supposing that the centre of gravity is on the axis (*a*) at the middle? (*b*) .25 meter from bottom, (*c*) the same distance from top?

XXVI. *Lever.* Articles 185–190; pp. 195–201.

[The weight of the lever is to be neglected, except when otherwise stated.]

1. The force $P = 40$ kilos acts as in Fig. 126, p. 196; $AF = 2$ meters and $AB = 3\frac{1}{2}$ meters: What weight can be supported?

2. If (Fig. 127, p. 196) $AB = 5$, $BF = 2\frac{1}{2}$ meters, and the weight is 60 kilos, what force P is required to support it?

3. If (Fig. 128, p. 196) $AB = 14$, $AF = 2$ meters, and $P = 24$ kilos, what weight can P support?

4. AFC is a bent lever (Fig. 129, p. 197); $AF = 12$, $FC = 14$, $AFC = 135°$, $P = 20$ kilos: What is W?

5. CFD is a bent lever (Fig. 131, p. 197); $CF = 16$, $FD = 24$, $PDF = 150°$, $FCW = 120°$, and $W = 60$ kilos: What is P?

6. A heavy uniform rod DF (Fig. 133, p. 199), weighing 10 kilos and $\frac{3}{4}$ meter long, is hinged at F; it is supported by a string carried from D to a point E, 250 milimeters vertically above F: What is the tension of the string?

7. A heavy uniform shelf DF (Fig. 134), $\frac{1}{2}$ meter wide, weighing 24 kilos, and hinged at F, is supported by a prop carried from C ($CF = 400$ millimeters) to a point E below F, so that $CF = FE$: What pressure does this prop feel?

8. Forces of 8 and 12 kilos act at the extremities of a straight bar 4 meters long, and in directions making angles of 120° and 150° respectively with it: Where is the fulcrum in case of equilibrium?

XXVII., XXVIII. *Balance—Steelyard.* Articles 191–196; pp. 202–210.

1. A body is equivalent to a weight of 6 kilos in one pan of a false balance, and of $6\frac{1}{2}$ kilos in the other: What is the true weight?

2. A body is equivalent to a weight of 84 grams from one arm of a false balance, and of 72 grams from the other: What is the ratio of the lengths of the arms?

3. The true weight of a body is 1 gram, its apparent weight in one pan of a balance is 960 milligrams: What would it seem to weigh in the other pan?

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4. The whole length of a steelyard is $\frac{3}{5}$ meter: CG (Fig. 138) $= 6$ millimeters, $CA = 20$ millimeters, $P = 250$ grams, and $Q = \frac{1}{2}$ kilo: (*a*) Where is the zero of the scale? (*b*) What is the length of graduation for 1 kilo? (*c*) How large weights can it be used for?

5. The weight of the beam of a steelyard is 1 kilo, and the distance of its centre of gravity is 16 millimeters from the fulcrum: Where must a counterpoise of 640 grams be placed to balance it?

6. The length of a Danish steelyard is $\frac{1}{2}$ meter, its weight is $1\frac{1}{2}$ kilos, and it acts at a point 50 millimeters from one end; a body weighing 6 kilos hangs at the other end: Where is the fulcrum?

XXIX., XXX. *Wheel and Axle—Pulley.* Articles 202–226; pp. 216–234.

1. The radius of the axle is 50 millimeters, that of the wheel is $\frac{3}{4}$ meter, and the power acting is 60 kilos: What weight is supported?

2. A man, exerting a force of 40 kilos on a lever-arm $1\frac{1}{4}$ meters long, turns a capstan; the radius of the circle about which the rope is wound is 150 millimeters: What is the pull felt upon the anchor?

3. A weight of 300 kilos hangs by a rope 20 millimeters in thickness; $r = .2$ meter, and $R = 1.25$ meters; the power acts on a lever-arm without a rope: What is P?

4. A power of 40 kilos balances a weight of 600 kilos; the radius of the axle is 75 millimeters: What is the diameter of the wheel?

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5. In a combination of pulleys, as in Fig. 162, $W = 704$ kilos, and $P = 44$ kilos: How many pulleys are there?

6. In a combination of pulleys, as in Fig. 163, $W = 288$ kilos and $P = 48$ kilos: How many pulleys are there?

7. In a combination as in Fig. 164, $W = 837$ kilos, $P = 27$ kilos: What is the number of pulleys?

8. In the single movable pulley, $W = 100$ kilos: Calculate the value of P when $2\alpha = 45°, = 135°$.

9. What force is needed to support 200 kilos by the second system of pulleys, there being 4 in all? What is the force if each pulley weighs $\frac{1}{2}$ kilo?

XXXI. *Inclined Plane.* Articles 227–232; pp. 235–241.

[The plane is supposed to be perfectly smooth.]

1. The angle of the plane is 30°, the weight is 120 kilos: What force is required to support the weight (*a*) acting parallel to the plane? (*b*) acting horizontally? (*c*) acting at an angle of 60° with the plane?

2. What is the reaction of the plane in the three cases in example 1?

3. A force of 60 kilos acts parallel to an inclined plane: What weight can it support in the following cases—the angle of the plane is (*a*) 20°, (*b*) 40°.

4. A weight of 60 kilos rests on an inclined plane: What force acting parallel to the plane is required to support it, if the angle of the plane has the same values as in example 3?

5. A force of 50 kilos acts horizontally to an inclined plane: What weight can it support if the angle of the plane is 45°?

6. If a horse can raise 800 kilos vertically, what weight can he raise on a railway having a grade of 44 feet to the mile?

XXXII, XXXIII. *Wedge—Screw.* Articles 233–242; pp. 243–256.

1. A wedge is isosceles in shape and has an angle of 30°; if $P = 20$ kilos, what is the resistance on each face?

2. A wedge is isosceles, and the angle 60°; a force of 100 kilos acts at the back: What are the other two forces?

3. A weight is isosceles, the power acting on the back is 20 kilos, and the forces on the other sides are 60 kilos each: What is the angle of the wedge?

4. The wedge is right-angled and the three sides have lengths of 25, 20, 15 (back): If $P = 100$, what are the other two forces?

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5. What weight can be raised by a power of 25 kilos acting on a lever-arm $\frac{3}{4}$ meter long, if the screw has 2 threads to the centimeter?

6. If a power of 40 kilos acting on an arm 1 meter long can support a weight of 8000 kilos, what will be the distance between the threads of the screw?

7. A screw has 1 thread to the centimeter; the circumference described by the power is 2 meters: What power is needed to support a weight of 3000 kilos?

8. While the point of application of the power makes a revolution of 1 meter, the screw advances 6 millimeters; the power is 50 kilos: What is the weight raised?

XXXIV. *Pendulum.* Articles 244–247; pp. 252–261.

[The length of the seconds pendulum at New York is about .9932 meter.]

1. Required the length of a pendulum at New York which will vibrate (*a*) in $\frac{1}{2}$ second, (*b*) in 3 seconds, (*c*) in $1\frac{1}{2}$ seconds.

2. Required the length of the seconds pendulum where the acceleration of gravity is 9.83 (that is, near the pole).

3. Required the length of a pendulum to vibrate in 2 seconds, where the value of g is 9.78.

4. How many beats in a minute would a pendulum $1\frac{1}{2}$ meters long make in New York?

ANSWERS TO EXAMPLES.

Pages 12, 13. I. ***Uniform Motion of Translation or Rotation.*** **Articles 17–21.**

(1) 490.91 miles. (2) 44 feet per second. (3*a*) $5\frac{13}{15}$ feet per second; (3*b*) $117\frac{1}{3}$ yards per minute. (4*a*) 3 miles; (4*b*) 25 seconds. (5) 1 mile. (6) 7.39 miles. (7) 19.01 miles.

(8) 1047.2 miles per hour, or 1535.9 feet per second. (9) 523.6 miles per hour, or 767.9 feet per second. (11) .000073. (12*a*) $\frac{4}{3}\pi = 4.189$; (12*b*) 12.57 feet per second. (13*a*) 628.3; (13*b*) 8567.9 miles. (14*a*) 5.236 feet per second; (14*b*) 10.472; (14*c*) 26.18.

Pages 21, 22. II. ***Uniformly Accelerated Motion.*** **Articles 22–28.**
A. Falling Bodies.

(1*a*) 480 ft. per sec.; (1*b*) 3600 ft.; (1*c*) 464 ft.; (1*d*) 1296 ft. (2*a*) 18 sec.; (2*b*) 576 ft. per sec. (3*a*) 16 sec.; (3*b*) 4096 ft. (4*a*) 11 sec.; (4*b*) 1936 ft. (5*a*) 12 sec.; (5*b*) 2304 ft. (6) $\frac{1}{4} : \frac{1}{2} : 1 : 3 : \frac{9}{2}$; the velocities are 8, 16, 32, 96, 144 ft. per sec. (7) $\frac{1}{16} : \frac{1}{4} : 1 : 9 : \frac{81}{4}$; the distances are 1 ft., 4, 16, 144, 324 feet. (8*a*) 31.46 sec.; (8*b*) 1006.86 ft. (9*a*) $55\frac{1}{9}$ ft.; (9*b*) $108\frac{4}{9}$. (10) 100 ft. (11*a*) 192 ft., 256 ft., 384 ft.; (11*b*) when B has fallen $5\frac{1}{2}$ sec. (12) 3.94 sec. (13) 576 ft.

B. General Case.—Acceleration $= f$.

(1) 8 ft.-per-sec. per sec. (2*a*) 20; (2*b*) 640 ft.; (2*c*) 160 ft. per sec. (3*a*) 72 ft. per sec.; (3*b*) 216 ft. (4) 8. (5*a*) 249.6 ft. per sec.; (5*b*) 374.4 ft. (6) $7\frac{1}{2}$ sec. (7) 6 sec. and 96 ft. (8) For the earth $v = 32, 64, 96$; and $s = 16, 48, 80$. The corresponding values for the sun are 28 times greater. (9) Yes. (10) Three times as far, *i.e.*, 150 ft.

Pages 30, 31. III. *Composition of Velocities.* Articles 29-37.

(1a) 1 mile per hour; (1b) 9 miles; (1c) 4 miles; (1d) 12 miles. (2a) At an angle of 38° 40′ with the line drawn directly across (BAC, Fig. 11); (2b) 6.4 miles per hour; (2c) 24 minutes; (2d) $1\frac{3}{5}$ miles down stream (BC, Fig. 11). (3a) 53° 8′ up stream (BAC, Fig. 12); (3b) 3 miles per hour; (3c) 40 minutes. (4a) At an angle of 73° 51′ up stream made with the line drawn directly across; (4b) 1.606 miles per hour; (4c) 1 hr. 26.3 min. (5a) 13° 51′ up stream measured from the line directly across; (5b) 5.61 miles per hour; (5c) 24.7 min. (6) Three fourths of a mile down stream. (7a) 13 ft. per sec.; (7b) N. 22° 37′ E. (8a) N. 22° 30′ E.; (8b) 14.78 ft. per sec. (9a) 15 ft. per sec.; (9b) 36° 52′ with his own direction. (10) 0. (11) 2.07 miles per hour due west. (12a) 128.06 yds. per minute at an angle 38° 40′ down stream; (12b) 960 yards below the starting-point (BC, Fig. 11); (12c) 53° 8′ up stream (BAC, Fig. 12); (12d) 20 minutes.

Pages 31, 32. IV. *Resolution of Constant Velocities.* Article 38.

(1) 7.79 ft. per sec. (2) 6, 10.39, 12, 10.39, 0. (3) 6.4 ft. per sec. parallel AC, and 4.8 parallel AB. (4) 5.2 miles per hour north, and 3 miles east. (5a) 8.66; (5b) 5. (6) 100 yds. per min. and at an angle of 53° 8′ up-stream. (7) 11.05 ft. per sec., $\alpha = 33° 34'$.

Pages 33, 34. V. *Falling down an Inclined Plane.* Article 40.

(1a) 16 ft.-per-sec. per sec.; (1b) 128 ft.; (1c) 64 ft. per sec.; (1d) 56 ft. (2a) 10 seconds; (2b) 80 ft. per sec. (3) 3.57 sec. (4a) 2 ft.-per-sec. per sec.; (4b) 36 feet. (5a) 4 ft.-per-sec. per sec.; (5b) 784 feet. (6a) 1024 feet; (6b) 128 ft. per sec. (7) The times are 5, $7\frac{1}{2}$, 10, 15, 20 sec.; the acquired velocity is 160 ft. per sec. in all.

Pages 39, 40. VI. *Bodies projected vertically downward.* Articles 41, 42.

(1a) 260 ft. per sec.; (1b) 1036 feet; (1c) 244 feet. (2a) $5\frac{1}{2}$ sec.; (2b) 196 ft. per sec. (3) 42 ft. per sec. (4) 75 ft. per sec. (5a) 55 ft. per sec.; (5b) 675 feet. (6) 21 ft. per sec. (7a) 64 feet; (7b) $3\frac{1}{2}$ sec. (8a) 1216 feet; (8b) 400 feet.

Pages 40, 41. VII. *Bodies projected vertically upward.* Article 44.

(1*a*, 1*b*) 9 sec; (1*c*) 1296 feet; (1*d*) 272 feet and 16 feet. (2*a*) 3 or 9 sec.; (2*b*) 15 or − 3 sec. (3*a*) 4½ sec.; (3*b*) 8½ sec.; (3*c*) 15 sec. (4) 240 ft. per sec. (5) 12 sec. (6) 192 ft. per sec. (7*a*) 112 ft. per sec.; (7*b*) same as (7*a*). (8) 256 ft. per sec. (9) 4 feet from the top after ½ second. (10*a*) 440 feet; (10*b*) 88 feet; (10*c*) 88 feet. (11) Two seconds after the second ball started. (13) It will actually ascend 3 seconds (though apparently falling as seen from the balloon) and then descend, reaching the ground after 7 seconds longer, or 10 seconds in all; the total distance, up and down, is 928 feet.

Pages 41, 42. VIII. *Projected up or down a smooth Inclined Plane.* Article 45.

(1*a*) 8 sec.; (1*b*) 89 feet. per sec. (2*a*) 96 ft. per sec.; (2*b*) 12 sec. (3*a*) 109 ft. per sec.; (3*b*) 308 feet; (3*c*) 101 feet. (4*a*) 5 sec.; (4*b*) 200 feet; (4*c*) 48 ft. per sec.; (4*d*) − 48.

Page 42. IX. *Bodies projected against Friction.* Articles 41, 42.

(1*a*) 12 ft.-per-sec. per sec.; (1*b*) after 10 sec.; (1*c*) 600 ft. (2*a*) 5 sec.; (2*b*) 100 feet; (2*c*) 16 ft. per sec. (3) 22 seconds and 242 feet. (4*a*) 60 miles per hour; (4*b*) four times as far. (5) 4000 feet; 20 seconds.

Page 50. X. *Projectiles.* Articles 47–51.

(1*a*) 5 seconds; (1*b*) 692.8 feet; (1*c*) 100 feet. (2) 2 or 3 seconds. (3) 7° 14′ or 82° 46′. (4) 367.65 ft. per sec. (5) 2½ seconds at a distance of 3000 feet. (6*a*) 1056 ft. per sec.; (6*b*) 400 feet. (7*a*) 120 feet; (7*b*) 48 and 80 ft. per sec.. (8*c*) after 1 second; (8*d*) 66 feet in a horizontal line from the point where it was dropped.

Page 55. XI. *Mass—Density—Volume.* Article 56.

(1) 1 : 4½. (2) 1½ : 1. (3) 9 : 8. (4) 2⅞ : 1. (5) .49 lb. (6) 1.21 : 1. (7) 1.1. (8) 1.024.

Page 63. XII. *Force of Gravity.* Articles 63–65.

(1) $4\sqrt{2}$ times the earth's radius. (2) g' (sun) : g (earth) $= \frac{350{,}000}{(112)^2} : 1$; that is, $g' = 27.9\,g$. (3) About ⅙ of g. (4) 317 : 1. (5) 32.047 (only the difference in distance from the earth's centre is considered).

Page 72. XIII. *Collision of Inelastic Bodies.* Article 70.

(1) $13\frac{1}{3}$ ft. per sec. (2) 8 ft. per sec. (3) 20 ft. per sec. (4) Respectively 8, 6, $4\frac{4}{5}$. (5) 4 : 1. (6) 8 : 7. (7) 7 : 5. (8) $29\frac{5}{8}$ ft. per sec. (9) 1631.76 ft. per sec.

Pages 80–82. XIV. *General Dynamical Problems.* Articles 68 and 73–76.

(1*a*) 4 ft.-per-sec. per sec.; (1*b*) 8 feet. (2*a*) At a distance equal to the earth's radius; (2*b*) one fourth as great as at the surface. (3) $5\frac{1}{3}$ ft.-per-sec. per second; the tension $(T) = 10$ lbs. (4) $\frac{1}{12}$ of g. (5*a*) 100 lbs.; (5*b*) 112.5. (6) 20 seconds. (7) $9\frac{3}{5}$ feet. (8) $1\frac{1}{15}$ oz. (9) $\frac{8}{11}$ lb. (10*a*) 27 lbs.; (10*b*) 21 lbs. (11) If the velocity is gained in 1 second, 40 lbs. (12*a*) $10\frac{2}{3}$ ft.-per-sec. per second; (12*b*) $85\frac{1}{3}$ feet. (13) $f = 20$; 15 oz. = 30 poundals. (14) $1\frac{10}{11}$ lbs. (15) $13\frac{3}{4}$ sec. (16) 300 lbs. (17) 96 lbs.

(18*a*) 8 sec.; (18*b*) 256 feet. (19) 5 lbs. (20*a*) $666\frac{2}{3}$ feet.; (20*b*) $8\frac{1}{3}$ sec. (21*a*) 215.4 ft. per sec.; (21*b*) 9.28 sec.

Pages 88, 89. XV. *Centripetal and Centrifugal Forces.* Articles 77–81.

(1) $f = 112$; tension = 70 lbs. (2*a*) They are increased 4 times; (2*b*) they are diminished one half. (3) 986.97 lbs. (4) 16 ft. per sec. (5) 2. (6) .726 ton. (7) 74.02 lbs. (8) 16.2 feet (only the tension caused by the circular motion is considered). (9) 1 foot.

Pages 99, 100. XVI. *Friction.* Articles 82–94

(1) $\mu = .25$. (2) 4.8 lbs. (3) 28 lbs. (4*a*) $\mu = .364$, $F = 5.13$ lbs.; (4*b*) $\mu = .364$, $F = 10.26$. (5) 4.657 lbs. (6*c*) 9.6 and 12 lbs. (7) $\mu = .54$. (8*a*) 6 lbs.; (8*b*) 5.638 lbs. (9*a*) $P = W \sin \alpha - F =$ 8.8 lbs.; (9*b*) $P = W \sin \alpha + F = 15.2$ lbs. (10) 10.35 lbs. (11) 13.05 lbs. (12) 208.46. (13) 167.42. (14) $6\frac{2}{3}$ lbs. (15) 12 lbs.

Pages 105, 106. XVII. *Work.* Articles 95–100.

(1) 125,028 ft.lbs. (2) 102,492.7 ft.lbs. (3) 1,440,000 ft.lbs. (4) 12,672,000 ft.lbs. (5) 5,940,000 ft.lbs. (6) 600,000 ft.lbs. (7) Against gravity 250,000 ft.lbs., against friction 180,000 ft.lbs.; total 430,000 ft.lbs. (8) Against gravity 60,000 ft.lbs., against

friction 24,000 (horizontal surface), 20,784.6 (inclined plane); total 104,784.6 ft.lbs.

Pages 124–126. XVIII. *Potential and Kinetic Energy.* Articles 101–108.

(1) 1800 ft.lbs. (2) 3,267,000,000 ft.lbs. (3) 9,375,000 ft.lbs. per minute, or 284.09 horse-power. (4) 250,000 ft.lbs. per minute, or 7.58 horse-power. (5) 6924.46 heat-units, and 0.111° C. (6) 4,500,000 ft.lbs., 3237.41 heat-units. (7) 625 ft.lbs. (8*a*) 10,000 feet; (8*b*) four times as far. (9) Same distance. (10*a*) 6400 feet; (10*b*) 40 sec; (10*c*) 624 feet; (10*d*) 304 ft. per sec. (11*a*) 5000 feet; (11*b*) 50 sec.; (11*c*) 18,000 ft.lbs. (12) 60.47 feet. (13) f (the retardation on the plane) $= \frac{4}{5}$ (32); (13*a*) 2000 feet; (13*b*) $12\frac{1}{2}$ sec. (14) 343.16 ft. per sec. (the weight of the body does not enter into the problem. (15*a*) $1360 \times W$ ft.lbs.; (15*b*) 6800 feet. (16) $R =$ 72 lbs. (17) 240,000 lbs. (18) 37,500 lbs.

Pages 142, 143. XIX. *Parallelogram of Forces.* Articles 126–136.

(1) 25 lbs., $\alpha = 73° 44'$. (2) 9.097, $\alpha = 30° 59'$. (3) $Q = 16.16$, $\gamma = 83° 29'$. (4) $\beta = 64° 39'$, $\gamma = 114° 9'$, $P = 5.94$; or $\beta = 16° 21'$, $\gamma = 65° 51'$, $P = 1.85$. (5) $\gamma = 120°$, $R = 27.71$ lbs. (6) 12.26 lbs. (7) 35 lbs., inclined 53° 8′ to the horizontal. (8) 20.78 lbs. (9) 15 and 20 lbs. (10) 100 lbs., 16° 16′ and 73° 44′. (11) 23.43 oz. (12) 5.72 and 11.44 lbs. (13) 41.57 lbs.

Pages 150, 151. XX. *Resolution of Forces.* Articles 137, 138.

(1) 106.05 lbs. N. and the same E. (2) 5 lbs., 5.18, 5.77, 7.07, 10, 19.30, ∞. (3) $a = 23.83$, $b = 31.11$ lbs. (4) $a = 34.64$ lbs., $b = 40$ lbs. (5) Each $= 39.16$ lbs. (6) 579.60 lbs. (7) 15.59 lbs. and 9 lbs.

Pages 151, 152. XXI. *Resolution of Forces along two rectangular axes.* Articles 140, 141.

(1) $R = 86.6$ lbs., and its direction makes an angle of 150° with P. (2) $R = 100$ lbs., at right angles to P. (3) $R = 208.3$ lbs., and makes an angle of 298° 41′ (or $-$ 61° 19′) with P. (4) $R = 73.2$ lbs., and acts due north. (5) $R = 346.4$ lbs., and acts S. 54° 44′ E. (6), the forces are in equilibrium. (7) $R = 61.22$ lbs., and acts S. 77° 30′ W.; the system will be kept in equilibrium by a force of 61.22 lbs acting N. 77° 30′ E.

Pages 159, 160. XXII. *Parallel Forces.* Articles 143–149.

(1*a*) $R = 12$, $AC = 28$, $BC = 20$; (1*b*) $R = 30$, $AB = 135$, $BC = 54$; (1*c*) $R = 6$, $BC = 40$, $AC = 16$; (1*d*) $R = 8$, $AB = 48$, $BC = 84$. (2*a*) $Q = 5$, $AC = 25$, $BC = 15$; (2*b*) $Q = 9$, $AC = 27$, $AB = 42$; (2*c*) $Q = -4$, $BC = 40$, $AC = 16$; (2*d*) $Q = -4$; $AB = 24$, $BC = 72$. (3) At A 36 lbs., at B 12 lbs. (4) W = 56 lbs., at C 24 lbs. (5) At B 9 lbs., at D 1 lb. (the weight of the rod is neglected). (6) A carries 84 lbs., B 60 lbs. (7) At A 4 lbs., at B and C 8 lbs. (8) At A 12 lbs., at B and C 3 lbs. (9) Placed at one end. (10) $31\frac{1}{2}$ and $76\frac{1}{2}$ lbs.

Page 167. XXIII. *Moments.* Articles 151–156.

(1) 72 ft.lbs. (2) 329.10 ft.lbs. (3) 615.64 ft.lbs. (4*a*) 72 ft.lbs. (4*b*) 55.15 ft.lbs.; (4*c*) 36 ft.lbs.

Pages 180–182. XXIV. *Centre of Gravity.* Articles 159–171.

(1) $13\frac{1}{3}$ in. from A. (2) 6 in. from C. (3) 15 in. from D. (4) 1 in. from B. (5) 2 in. from B. (6) $2\frac{1}{2}$ lbs. (7) 4 oz. (8) $\frac{1}{2}$ lb. (9*a*) A carries 24 lbs., B 36 lbs.; (9*b*) A should stand 8 feet from his end. (10) $1\frac{1}{11}$ in. from the centre toward A. (11) 6 in. from C, on a line making an angle of 53° 8′ with BC. (12) 16 in. from A, on a line bisecting the angle BAC. (13) On a line bisecting the right angle, $3\sqrt{2}$ in. from B. (14*a*) $3\sqrt{2}$ in. from the centre toward A; (14*b*) $4\frac{4}{5}$ in. from the centre on a line drawn to the middle point of AB; (14*c*) $2\sqrt{2}$ from the centre toward B. (15*a*) $2\frac{2}{3}$ in. from the centre, on the line drawn to the middle point of CD; (15*b*) $2\sqrt{2}$ in. from the centre toward C; (15*c*) $\frac{5}{7}\sqrt{2}$ from the centre toward C. (16) $2\frac{2}{3}$ in. from the middle point of BC, on the line joining the centres of the parallel sides. (17) 1 in. from the centre of the original circle. (18*a*) $2\frac{4}{7}$ in. from the centre of the larger circle. (19) $3\frac{1}{7}$ in. from the centre of the base. (20) 8 in. from B, on the line BD. (21) $11\frac{1}{3}$ in. from the vertex of the triangle. (22) On the axis, 4 in. from the centre of the larger cylinder.

Pages 189, 190. XXV. *Stability.* Articles 172–177.

(1*a*) 45 lbs.; (1*b*) 90 lbs.; (1*c*) 125 lbs. and 250 lbs. (2*a*) 42.43 lbs.; (2*b*) 43.92 lbs. (3) 48° 11′. (4) 80 feet vertically. (5*a*) 45°;

(5*b*) 67° 10′ (the vertex lying up the plane). (6*a*) 16 lbs.; (6*b*) 48 lbs. (7*a*) $P=8$ lbs.; (7*b*) 20 lbs.; (7*c*) $6\frac{2}{3}$ lbs. (8*a*) 15,406.6 ft.lbs.; (8*b*) 49,442.7 ft.lbs.; (8*c*) 8781.8 ft.lbs.

Pages 213, 214. XXVI. *Lever.* Articles 185–190.

(1) 160 lbs. (2) 20 lbs. (3) $12\frac{1}{2}$ lbs. (4) 200 lbs., 100 lbs., $87\frac{1}{2}$ lbs. respectively. (5) 37.12 lbs. (6) 46.58 lbs. (7) 180 lbs. (8) 67.12 vertical (5), and 163.71 lbs. in a direction making an angle of 12° 13′ with a vertical line through F (7). (9) 29.15 lbs. (10) 38.18 lbs. (11) 96 and 48 lbs. (12) $\frac{1}{8}$ of the length from the centre toward the end having the heavier weight. (13) 8.24 feet from the end at which the force 8 acts.

Page 214. XXVII. *Balance.* Articles 191–193.

(1) 14.07 lbs. (2) .837 : 1. (3) 14.06 oz.

Page 215. XXVIII. *Steelyard.* Articles 194–196.

(1*a*) The zero is $\frac{3}{8}$ inch from $C\,(=CD$, Fig. 139); (1*b*) $\frac{3}{8}$ in. from $C\,(=CG$, Fig. 139); (1*c*) the graduation is to 12ths of an inch; (1*d*) 1 lb. and 20 lbs. (2*a*) $\frac{1}{3}$ in. from $C\,(=CD$, Fig. 138); (2*b*) $1\frac{1}{3}$ in. for 1 lb.; (2*c*) $15\frac{1}{6}$ lbs. (3) $\frac{1}{4}$ in. and $\frac{1}{12}$ in. (4) $\frac{3}{7}$ in. from the fulcrum. (5) $6\frac{2}{4}$ from the end on which hangs the weight. (6*a*) 18 in. from the end A (Fig. 141); (6*b*) $14\frac{2}{5}$ in. from B.

Page 220. XXIX. *Wheel and Axle.* Articles 202–208.

(1) $W=1200$ lbs. (2) 21,600 lbs. (3) 1920 lbs. (4) $P=88.54$ lbs. (5) 8 feet 4 in. (6) $P=10.61$.

Pages 234, 235. XXX. *Pulley.* Articles 217–226.

(1*a*) 50 lbs. (1*b*) 150 lbs. (the weight of the platform is neglected in each case). (2) $n=4$. (3) $n=4$. (4) $n=4$. (5) $W=193.18$ (30°), $=173.2$ (60°), $=100$ (120°), $=51.76$ (150°), $=0$ (180°). (6) $P=31.25$, and 31.72 lbs. (7) $P=125$ and 125.25. (8) $P=33.33$ and 32.97. (9) If w' is the weight of the pulley A, and w'' that of B, then $4P+w''=W+w'$. (10) Let the weights of the movable pulleys D, C, B, A be respectively $w', w'', w''', w^{\text{iv}}$, then $P=\frac{W}{81}+\frac{w'}{81}+$

$\frac{w''}{27}+\frac{w'''}{9}+\frac{w^{\text{iv}}}{3}$. If the weights of the pulleys (w) are equal, then $P=\frac{W}{3^4}-\frac{w}{2}\left(1-\frac{1}{3^4}\right)$; or, in general, $P=\frac{W}{3^n}-\frac{w}{2}\left(1-\frac{1}{3^n}\right)$.

Pages 241–243. XXXI. *Inclined Plane.* Articles 227–232.

(1*a*) $P=41.04$; (1*b*) $P=43.68$; (1*c*) $P=47.39$. (2*a*) $R=112.76$; (2*b*) $R=127.7$; (2*c*) $R=89.07$. (3*a*) 575.9; (3*b*) 200; (3*c*) 141.4; (3*d*) 115.5; (3*e*) 101.5; (3*f*) 100. (4*a*) 17.36; (4*b*) 50; (4*c*) 70.7; (4*d*) 86.6; (4*e*) 98.5; (4*f*) 100. (5*a*) 567.1; (5*b*) 173.2; (5*c*) 100; (5*d*) 57.7; (5*e*) 17.6; (5*f*) 0. (6*a*) 17.6; (6*b*) 57.7; (6*c*) 100; (6*d*) 173.2; (6*e*) 567.1; (6*f*) ∞. (7) $\frac{5}{12}$ of a ton. (8) 35 and 21 lbs. (9) 11,464. lbs. (10) $\frac{1}{120}$ of the weight. (11) $\alpha=36^\circ\ 52'$, $W=66\frac{2}{3}$. (12) $5\frac{1}{3}$. (13) 12 feet and 18 feet.

Page 232. XXXII. *Wedge.* Articles 233–235.

(1) 115.2. (2) 70.7. (3) $38^\circ\ 56'$. (4) They are equal. (5) $133\frac{1}{3}$ and $166\frac{2}{3}$. (6) $1 : 2 : \sqrt{3}$.

Page 251. XXXIII. *Screw.* Articles 236–242.

(1) 9047.8 lbs. (2) .785 in. (3) $12\frac{1}{2}$ lbs. (4) 7200 lbs. (5) 1 : 113.4. (6) 2. (7) $\frac{3}{4}$ in. (8) 81,430.3 lbs.

Page 261. XXXIV. *Pendulum.* Articles 244–247.

(1*a*) 9.78 in.; (1*b*) 13.03 feet; (1*c*) 20.37 feet. (2) 39.21 in. (3) 13.01 feet. (4) About 91.2 feet. (5) 38.3 times. (6) 37.5 in.

ANSWERS TO ADDITIONAL EXAMPLES

(ON PAGES 263–278)

INTRODUCING THE METRIC UNITS.

I.

(1) 864 kilometers. (2) $13\frac{8}{9}$. (3a) $16\frac{2}{3}$; (3b) 100. (4a) 5.44 kilometers; (4b) $8\frac{14}{17}$ minutes. (5) 2.56 kilometers. (6a) 88 (per second); (6b) 3168 kilometers.

II., A.

(1) The distances are 4.9, 19.6, 44.1, 78.4 meters; the velocities are 9.8, 19.6, 29.4, 39.2 meters per second. (2a) 98 meters per second; (2b) 490 meters; (2c) 93.1 meters. (3a) $4\frac{2}{7}$ seconds. (3b) 42 meters per second. (4a) $7\frac{1}{2}$ minutes; (4b) 275.6 meters. (5a) 5 seconds; (5b) 122.5 seconds.

II., B.

(1) 8 meters-per-second per second. (2a) 20 meters-per-second per second; (2b) .36 kilometers; (2c) 120 meters per second. (3a) 60 meters per second; (3b) 150 meters. (4) Yes. (5) 260 meters. (6) 20 meters-per-second per second.

III.

(1a) 2 kilometers per hour; (1b) 18; (1c, 1d) 8 and 24. (2a) 38° 40′ down stream; (2b) 128.1 meters per minute; (2c) 20 minutes. (3a) 53° 8′ up stream; (3b) 60 meters per minute; (3c) $33\frac{1}{3}$ minutes. (4a) 13 meters per second; (4b) N. 22° 37′ E. (5a) N. 67° 30′ E.; (5b) 6.12 meters per second. (6a) 5 meters per second; (6b) 36° 52′ with his direction.

IV.

(1) 3 meters per second. (2) $4\frac{4}{5}$ and $3\frac{3}{5}$ meters per second. (3) 4.33 and 2.5. (4) 12.99 and 7.5.

V.

(1*a*) 4.9 meters-per-second per second; (1*b*) 39.2; (1*c*) 19.6. (2*a*) 8 seconds; (2*b*) 19.6 meters per second. (3) 11.89 seconds. (4*a*) 1.4 meters-per-second per second; (4*b*) 90 meters.

VI.

(1*a*) 80.6 meters per second; (1*b*) 324.1 meters. (2*a*) 5 seconds; (2*b*) 68.1 meters per second. (3) 18.63 meters per second. (4*a*) 19.6 meters; (4*b*) $3\frac{4}{7}$ seconds.

VII.

(1*a*) and (1*b*) 5 seconds; (1*c*) 122.5 meters; (1*d*) 44.1 and 4.9 meters. (2) 1 or $7\frac{4}{7}$ seconds. (3) 56 meters per second. (4) $7\frac{1}{7}$ seconds. (5*a*) 28 meters per second; (5*b*) after $1\frac{5}{7}$ seconds longer.

VIII.

(1*a*) 12 seconds ; (1*b*) 49.7 meters per second. (2*a*) 29.6 meters per second ; (2*b*) 79.2 meters. (3*a*) 56 meters per second ; (3*b*) $11\frac{3}{7}$ seconds.

IX.

(1*a*) $f = 4$ meters-per-second per second; (1*b*, 1*c*) after sliding 10 seconds, and 200 meters. (2*a*) 20 seconds ; (2*b*) 120 meters; (2*c*) 10.2 meters per second. (3*a*) 28.8 kil. per hour ; (3*b*) 480 meters.

X.

(1*a*) $t = 25$ seconds; (1*b*) 5.304 kilometers. (2) $24° 18'$ or $65° 42'$. (3) $u = 280$ meters per second. (4) After 3 seconds, at a distance of 1.2 kilometers. (5*a*) 200 meters per second; (5*b*) 122.5 meters.

XI.

(1) $2\frac{2}{3}$: 1. (2) 2.73 : 1. (3) 773.2. (4) 13.394 kilograms.

XII.

(1) 4 times the earth's radius. (2) 277.3 meters-per-second per

second. (3) About 1.6 meters-per-second per second. (4) 2.7 millimeters-per-second per second.

XIII.

(1) 5 meters per second. (2) 3 meters per second. (3) 10 meters per second. (4) 4, 3, and 2.4 meters per second. (5) 420.84 meters per second.

XIV.

(1*a*) 3.8 meters-per-second per second; (1*b*) 7.6 meters. (2*a*) At a height equal to 1$\frac{4}{5}$ the radius of the earth; (2*b*) 2.04 kilos. (3) $f = 2.8$; tension, 4.286 kilos (4) 20.4 seconds. (5) 28$\frac{4}{7}$ kilos. (6*a*) 400 grams; (6*b*) 392,000 dynes. (7) 1.4 grams. (8) 1.02 milligrams.

XV.

(1) 25 kilos; $f = 24.5$. (2) 8297$\frac{1}{7}$ kilos.

XVI.

(1) $\mu = \frac{1}{3}$. (2) 4.8 kilos. (3) 160 grams. (4*a*) $\mu = .36$, $F = 5.07$ kilos. (4*b*) $\mu = .36$, $F = 10.14$. (5) 6.235 kilos.

XVII.

(1) 1 kilogram-meter = 7.23 ft.lbs. (2) 62,514 kilogram-meters. (3) 675,000 kilogram-meters. (4) 100,000 kilogram-meters. (5) 172,000 kilogram-meters.

XVIII.

(1) About 424 kilogram meters. (2) 200 kilogram-meters. (3) 50,000,000 kilogram-meters. (4) 400,000 kilogram-meters. (5) 270 kilogram-meters. (6) 4 kilometers and 57$\frac{1}{7}$ seconds. (7) 50 kilos. (8) 150,000 kilos.

XIX.

(1) $R = 250$ grams; (α) 73° 44′. (2) 11.14. (3) $R = 20.78$, $\gamma = 120°$. (4) 6.13 kilos. (5) 910 grams. (6) 8 kilos.

XX., XXI.

(1) 56.4 N. 20.52 E. (2) 6.93 and 13.86. (3) 25.21 kilos. (4) 295.44 kilos. (5) 554.24 and 320 grams.

(1) $R = 303.5$, the angle between P and R is $-10° 46'$. (2) 100 kilos in the same direction as P.

XXII.

(1) 16 kilos at A, and 8 at B. (2) $10\frac{1}{2}$ at C; $W = 24\frac{1}{2}$. (3) A carries 48 and B 32 kilos. (4) On A 2 kilos, and on B and C each 4 kilos. (5) 2 centimeters from one end.

XXIII.

(1) 72 kilogram-meters. (2) 30 kilogram-meters: the same as before. (3) 346.4 kilogram-meters. (4) 25, 23.5, and 8.55 kilogram-meters.

XXIV., XXV.

(1) 21 centimeters from A. (2) 25 millimeters from B. (3) 270 grams. (4) 10.02 gr. (5) 125 from the centre toward A. (6) 69 millimeters from A, on the line drawn to the middle point of BC. (7a) 21 kilos; (7b) 56 kilos. (8a) 7.54 kilos; (8b) 38.63 kilos; (8c) 6.63 kilos. (9a) 811.4 kilogram-meters; (9b) 3090.2; (9c) 450.85.

XXVI.

(1) $53\frac{1}{3}$ kilos. (2) 20 kilos. (3) 3 kilos. (4) 24.24 kilos. (5) 69.28 kilos. (6) 15.81 kilos. (7) 21.21 kilos. (8) 1 meter 856.4 mm. from the force 8.

XXVII., XXVIII.

(1) 6 kilos 245 grams. (2) 1 : 1.08. (3) 1041.7 milligrams. (4a) 12 millimeters from C; (4b) 80 millimeters; (4c) 7.1 kilos. (5) 25 millimeters from the fulcrum. (6) 90 millimeters from the end on which hangs the weight.

XXIX, XXX.

(1) 900 kilos. (2) $333\frac{1}{3}$ kilos. (3) 50.4 kilos. (4) $2\frac{1}{4}$ meters. —(5) 4 pulleys. (6) 6 pulleys. (7) 5 pulleys. (8) 54.12 ($2\alpha = 45°$), 130.66 ($2\alpha = 135°$). (9) 50 kilos, 50.25 kilos.

XXXI.

(1a) 60 kilos; (1b) 69.28 kilos; (1c) 120 kilos. (2a) 103.92; (2b)

138.56; (2*c*) 0. (3*a*) 175.4; (3*b*) 93.34. (4*a*) 20.52; (4*b*) 38.57. (5) 50 kilos. (6) 96,000 kilos.

XXXII., XXXIII.

(1) 38.64. (2) Each 100 kilos. (3) 19° 12′. (4) 100, $133\frac{1}{3}$, $166\frac{2}{3}$.—(5) 25,132.7 kilos. (6) 31.42 millimeters. (7) 15 kilos. (8) $8333\frac{1}{3}$ kilos.

XXXIV.

(1*a*) .110 meter; (1*b*) 8.939 meters; (1*c*) 2.235 meters. (2) .9962 meters. (3) 3.965 meters. (4) 48.8.